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Finite Fuzzy Description Logics and Crisp Representations

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Abstract. Fuzzy Description Logics (DLs) are a formalism for the representation of structured knowledge that is imprecise or vague by nature. In fuzzy DLs, restricting to a finite set of degrees of truth has proved to be useful, both for theoretical and practical reasons. In this paper, we propose finite fuzzy DLs as a generalization of existing approaches. We assume a finite totally ordered set of linguistic terms or labels, which is very useful in practice since expert knowledge is usually expressed using linguistic terms. Then, we consider fuzzy DLs based on any smooth t-norm defined over this set. Initially we focus on the finite fuzzy DL $\mathcal{ALCH}$, studying some logical properties, and showing the decidability of the logic by presenting a reasoning preserving reduction to the classical case. Finally, we extend our logic in two directions: by considering non-smooth t-norms and by considering additional DL constructors.

1 Introduction

It has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real-world domains. Since fuzzy logic is a suitable formalism to handle these types of knowledge, there has been an important interest in generalizing ontologies to the fuzzy case. Description Logics (DLs) are a family of logics for representing structured knowledge [1], and many ontology languages are based on DLs [2]. Because of the need of managing imprecise and vagueness, several fuzzy DLs can be found in the literature. For a good survey, we refer the reader to [3]. Notice that the extension of ontologies and DLs with other formalisms to deal with imprecision and vagueness, such as rough set theory, has also been studied [4,5,6,7].

It is well known that different families of fuzzy operators (or fuzzy logics) lead to fuzzy DLs with different properties. For example, Gödel and Zadeh fuzzy logics have an idempotent conjunction, whereas Łukasiewicz and Product fuzzy logic do not. Clearly, different applications may need different fuzzy logics. For example, Łukasiewicz logic may not be suitable for combining information, as the conjunction easily collapses to zero [8].

Some recent results show that some fuzzy DLs with infinite model property are undecidable [9,11,10]. Also, in fuzzy DLs the infinite model property does not hold in relatively non expressive fuzzy DLs [12,13]. This makes the study of finite fuzzy DLs even more interesting.

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In fuzzy DLs, assuming a finite set of degrees of truth is useful [14,15,16]. In Zadeh fuzzy logic it is interesting for computational reasons [14]. In Gödel logic, it is necessary to show that the logic satisfies the Witnessed Model Property [12]. In Łukasiewicz logic, it is necessary to obtain a classical representation of the fuzzy ontology [16]. The objective of our research is to study whether a finite set of degrees of truth we can assumed when fuzzy logics different to Zadeh, Gödel, and Łukasiewicz are considered. As we will see, the answer is positive.

There is a recent promising line of research that tries to fill the gap between mathematical fuzzy logic and fuzzy DLs [8,12,17,18,19]. Following this path, we build on the previous research on finite fuzzy logics [20,21,22,23] and propose a generalization of the existing approaches to fuzzy DLs under finite degrees of truth that have been proposed in the literature [14,15,16].

Instead of dealing with degrees of truth in $[0, 1]$, as usual in fuzzy DLs, we will assume a finite chain (a finite totally ordered set) of linguistic terms or labels. For instance, $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$. Then, we will start by considering any smooth t-norm defined over a chain of degrees of truth. Later on, we will also consider the non-smooth case.

In summary, the contributions of this paper are two-fold. On the one hand, we study the use of a finite chain of labels in fuzzy ontologies. This makes it possible to abstract from the numerical interpretations of these labels. This way, since experts' knowledge is usually expressed using a set of linguistic terms [20], the process of knowledge acquisition is easier. On other hand, we consider the general case of finite fuzzy DLs, starting from a finite smooth t-norm but also discussing the case of non-smooth t-norms. This makes it possible to use new fuzzy operators (e.g., QL-implications) in fuzzy DLs for the first time.

The use of linguistic labels as degrees in fuzzy DLs has already been proposed. [24] proposes to take the degrees from an uncertainty lattice. A recent extension of this work by other authors considers Zadeh $\mathcal{SHIN}$ [25]. Finite chains of degrees of truth have also been considered in the setting of fuzzy DLs. In [18,19] the authors use them as one of the building blocks of the first order t-norm based logic $L^*_\sim(\mathbf{S})\forall$, which can be used to define several related fuzzy DLs starting from a t-norm $*$. The difference with our work is that we directly consider fuzzy DLs and hence are to able to provide specific reasoning algorithms.

The remainder is organized as follows. Section 2 includes some preliminaries on finite fuzzy logics and classical DLs. Then, Section 3 defines a fuzzy extension of the DL $\mathcal{ALCH}$ based on finite fuzzy logics, discusses some logical properties, and shows the decidability of the logic by providing a reduction of fuzzy $\mathcal{ALCH}$ into crisp $\mathcal{ALCH}$. Section 4 discusses some extensions of this logic obtained by considering non-smooth t-norms or other DL constructors. Finally, Section 5 sets out some conclusions and ideas for future research.

2 Preliminaries

This section is split into two parts. Section 2.1 reviews some results about finite fuzzy logics, and Section 2.2 overviews the classical DL $\mathcal{ALCH}$.

2.1 Finite Fuzzy Logics

Fuzzy set theory and fuzzy logic were proposed by L. Zadeh [26] to manage imprecise and vague knowledge. Here, statements are not either true or false, but they are a matter of degree. Let X be a set of elements called the reference set, and let $\mathcal{S}$ be a totally ordered set with e as minimum element and u as maximum. A *fuzzy subset* A of X is defined by a membership function $A(x) : X \to \mathcal{S}$ which assigns any $x \in X$ to a value in $\mathcal{S}$. Similarly as in the classical case, e means no-membership and u full membership, but now a value between them represents to which extent x can be considered as an element of X.

In the following, we restrict to finite chains of degrees of truth. The rest of this section contains material from [20,21,22,23].

Definition 1. *A* finite chain *of degrees of truth is a totally ordered set* $\mathcal{N} = \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_p = 1\}$*, where* $p \geq 1$*.*

Example 1. {`false`, `closeToFalse`, `weaklyFalse`, `weaklyTrue`, `closeToTrue`, `true`} is a finite chain.

$\mathcal{N}$ can be understood as a set of linguistic terms or labels. For our purposes all finite chains with the same number of elements are equivalent.

In the rest of the paper, we will use the following notation: $\mathcal{N}^+ = \mathcal{N} \setminus \{\gamma_0\}$, $+\gamma_i = \gamma_{i+1}$, $-\gamma_i = \gamma_{i-1}$. Let us also denote by $[\gamma_i, \gamma_j]$ the finite chain given by the subinterval of all $\gamma_k \in \mathcal{N}$ such that $i \leq k \leq j$.

All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication are performed by a t-norm function, a t-conorm function, a negation function, and an implication function, respectively. These functions can be restricted to finite chains. Table 1 shows some popular examples: Zadeh, Gödel, and Łukasiewicz.

Definition 2. *A* t-norm *on* $\mathcal{N}$ *is a function* $\otimes : \mathcal{N}^2 \to \mathcal{N}$ *such that for all* $\gamma_i, \gamma_j, \gamma_k \in \mathcal{N}$ *the following conditions are satisfied:*

- $\gamma_i \otimes \gamma_j = \gamma_j \otimes \gamma_i$,
- $(\gamma_i \otimes \gamma_j) \otimes \gamma_k = \gamma_i \otimes (\gamma_j \otimes \gamma_k)$,
- $(\gamma_i \otimes \gamma_j) \leq (\gamma_i \otimes \gamma_k)$ *whenever* $\gamma_j \leq \gamma_k$,
- $\gamma_i \otimes \gamma_p = \gamma_i$.

Definition 3. *A function* $f : \mathcal{N} \to \mathcal{N}$ *is* smooth *iff it satisfies the following condition for all* $i \in \mathcal{N}^+$ $f(\gamma_i) = \gamma_j$ *implies that* $f(\gamma_{i-1}) = \gamma_k$ *with* $j - 1 \leq k \leq j + 1$*. A binary operator is smooth when it is smooth in each place.*

The *smoothness condition* is a discrete counterpart of continuity on $[0, 1]$. Smoothness for t-norms is equivalent to the divisibility condition in $[0, 1]$, i.e., $\gamma_i \leq \gamma_j$ if and only if there exists $\gamma_k \in \mathcal{N}$ such that $\gamma_j \otimes \gamma_k = \gamma_i$.

Definition 4. *A t-norm* $\otimes$ *is* Archimedean *iff* $\forall \gamma_1, \gamma_2 \in \mathcal{N} \setminus \{\gamma_0, \gamma_p\}$ *there is* $n \in \mathbb{N}$ *such that* $\gamma_1 \otimes \gamma_1 \cdots \otimes \gamma_1$ *(n times)* $< \gamma_2$*.*

Table 1. Popular fuzzy logics over a finite chain

Family	$\gamma_i \otimes \gamma_j$	$\gamma_i \oplus \gamma_j$	$\ominus\gamma_i$	$\gamma_i \Rightarrow \gamma_j$
Zadeh	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	γ_{p-i}	$\max\{\gamma_{p-i}, \gamma_j\}$
Gödel	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	$\begin{cases} \gamma_p, \gamma_i = 0 \\ \gamma_0, \gamma_i > 0 \end{cases}$	$\begin{cases} \gamma_p, \gamma_i \leq \gamma_j \\ \gamma_j, \gamma_i > \gamma_j \end{cases}$
Łukasiewicz	$\gamma_{\max\{i+j-p,0\}}$	$\gamma_{\min\{i+j,p\}}$	γ_{p-i}	$\gamma_{\min\{p-i+j,p\}}$

Proposition 1. *There is one and only one Archimedean smooth t-norm on $\mathcal{N}$ given by $\gamma_i \otimes \gamma_j = \gamma_{\max\{0,i+j-p\}}$. Moreover, given any subset J of $\mathcal{N}$ containing γ_0, γ_p, there is one and only one smooth t-norm $\otimes^J$ on $\mathcal{N}$ that has J as the set of idempotent elements* [1]. *In fact, if J is the set $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$ such a t-norm is given by:*

$$\gamma_i \otimes^J \gamma_j = \begin{cases} \gamma_{\max\{i_k, i+j-i_{k+1}\}} & \text{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \text{ for some } 0 \leq k \leq m-1 \\ \gamma_{\min\{i,j\}} & \text{otherwise} . \end{cases}$$

Notice that the Archimedean smooth t-norm is obtained when $J = \{\gamma_0, \gamma_p\}$, and that the minimum is obtained when $J = \mathcal{N}$. It is also worth to note that, as a consequence of Proposition 1, a finite smooth product t-norm is not possible.

Example 2. Given the finite chain $\mathcal{N} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ and the set $J = \{\gamma_0, \gamma_3, \gamma_5\}$, $\otimes^J$ is defined as:

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_0	γ_0	γ_1	γ_1	γ_1
γ_2	γ_0	γ_0	γ_1	γ_2	γ_2	γ_2
γ_3	γ_0	γ_1	γ_2	γ_3	γ_3	γ_3
γ_4	γ_0	γ_1	γ_2	γ_3	γ_3	γ_4
γ_5	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5

Definition 5. *A* strong negation *on $\mathcal{N}$ is a function $\ominus : \mathcal{N} \to \mathcal{N}$ such that for all $\gamma_i, \gamma_j \in \mathcal{N}$ the following conditions are satisfied:*

- $\gamma_i < \gamma_j$ *implies* $\ominus\gamma_i > \ominus\gamma_j$,
- $\ominus\gamma_0 = \gamma_p, \ominus\gamma_p = \gamma_0$,
- $\ominus(\ominus\gamma_i) = \gamma_i$ *for all* $\gamma_i \in \mathcal{N}$.

There is only one strong negation on $\mathcal{N}$ and it is given by $\ominus\gamma_i = \gamma_{p-i}$

Definition 6. *A* t-conorm *on $\mathcal{N}$ is a function $\oplus : \mathcal{N}^2 \to \mathcal{N}$ such that for all $\gamma_i, \gamma_j, \gamma_k \in \mathcal{N}$ the following conditions are satisfied:*

- $\gamma_i \oplus \gamma_j = \gamma_j \oplus \gamma_i$,
- $(\gamma_i \oplus \gamma_j) \oplus \gamma_k = \gamma_i \oplus (\gamma_j \oplus \gamma_k)$,
- $(\gamma_i \oplus \gamma_j) \leq (\gamma_i \oplus \gamma_k)$ *whenever* $\gamma_j \leq \gamma_k$,
- $\gamma_i \oplus \gamma_0 = \gamma_i$.

[1] γ is idempotent iff $\gamma \otimes \gamma = \gamma$.

Proposition 2. *There is one and only one Archimedean smooth t-conorm on* $\mathcal{N}$ *given by* $\gamma_i \oplus \gamma_j = \gamma_{\min\{p,i+j\}}$. *Moreover, given any subset* J *of* $\mathcal{N}$ *containing* γ_0, γ_p, *there is one and only one smooth t-conorm* $\oplus^J$ *on* $\mathcal{N}$ *that has* J *as the set of idempotent elements. In fact, if* J *is the set* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$ *such a t-conorm is given by:*

$$\gamma_i \oplus^J \gamma_j = \begin{cases} \gamma_{\min\{i_{k+1}, i+j-i_k\}} & \text{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \text{ for some } 0 \leq k \leq m-1 \\ \gamma_{\max\{i,j\}} & \text{otherwise .} \end{cases}$$

Note that the Archimedean smooth t-conorm is obtained when $J = \{\gamma_0, \gamma_p\}$, and that the maximum is obtained when $J = \mathcal{N}$.

Given a t-norm $\otimes$ and the strong negation $\ominus$, we can define the *dual* t-conorm $\oplus_{\otimes}$, as the function satisfying $\gamma_i \oplus \gamma_j = \ominus((\ominus\gamma_i) \otimes (\ominus\gamma_j))$.

Definition 7. *A binary operator* $\Rightarrow: \mathcal{N}^2 \rightarrow \mathcal{N}$ *is said to be an* implication, *if the following conditions are satisfied:*

- *if* $\gamma_i \leq \gamma_j$ *then* $(\gamma_i \Rightarrow \gamma_k) \geq (\gamma_j \Rightarrow \gamma_k)$ *for all* $\gamma_k \in \mathcal{N}$,
- *if* $\gamma_i \leq \gamma_j$ *then* $(\gamma_k \Rightarrow \gamma_i) \leq (\gamma_k \Rightarrow \gamma_j)$ *for all* $\gamma_k \in \mathcal{N}$,
- $\gamma_0 \Rightarrow \gamma_0 = \gamma_p \Rightarrow \gamma_p = \gamma_p$ *and* $\gamma_p \Rightarrow \gamma_0 = \gamma_0$.

Definition 8. *Given a t-norm* $\otimes$ *and the strong negation* $\ominus$, *an* S-implication $\Rightarrow_{s\otimes}$ *is the function satisfying* $\gamma_i \Rightarrow_{s\otimes} \gamma_j = \ominus(\gamma_i \otimes (\ominus\gamma_j))$.

Equivalently, an S-implication can also be defined as $\gamma_i \Rightarrow_{s\otimes} \gamma_j = (\ominus\gamma_i) \oplus \gamma_j$.

Proposition 3. *Let* $\otimes^J : \mathcal{N}^2 \rightarrow \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$. *Then, the implication* $\Rightarrow_{s\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{s\otimes} \gamma_j = \begin{cases} \gamma_{\min\{p-i_k, i_{k+1}+j-i\}} & \text{if } \exists \gamma_{i_k} \in J \text{ such that } \gamma_{i_k} \leq \gamma_i, \gamma_{p-j} \leq \gamma_{i_{k+1}} \\ \gamma_{\max\{p-i,j\}} & \text{otherwise .} \end{cases}$$

The Łukasiewicz implication is obtained for the Archimedean t-norm. Similarly, the Kleene-Dienes implication $\gamma_i \Rightarrow \gamma_j = \max\{\gamma_{p-i}, \gamma_j\}$ is obtained for the minimum t-norm. This is the reason why we refer to the corresponding fuzzy logic that includes Kleene-Dienes implication as Zadeh fuzzy logic.

Definition 9. *Given a t-norm* $\otimes$, *an* R-implication $\Rightarrow_{r\otimes}$ *can be defined as* $\gamma_i \Rightarrow_{r\otimes} \gamma_j = \max\{\gamma_k \in \mathcal{N} \mid (\gamma_i \otimes \gamma_k) \leq \gamma_j\}$, *for all* $\gamma_i, \gamma_j \in \mathcal{N}$.

Proposition 4. *Let* $\otimes^J : \mathcal{N}^2 \rightarrow \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$. *Then, the implication* $\Rightarrow_{r\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{r\otimes} \gamma_j = \begin{cases} \gamma_p & \text{if } \gamma_i \leq \gamma_j \\ \gamma_{i_{k+1}+j-i} & \text{if } \exists \gamma_{i_k} \in J \text{ such that } \gamma_{i_k} \leq \gamma_j < \gamma_i \leq \gamma_{i_{k+1}} \\ \gamma_j & \text{otherwise .} \end{cases}$$

Example 3. Given the t-norm in Example 2, $\Rightarrow_{r\otimes}$ is defined as follows, where the first column is the antecedent and the first row is the consequent:

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
γ_0	γ_5	γ_5	γ_5	γ_5	γ_5	γ_5
γ_1	γ_2	γ_5	γ_5	γ_5	γ_5	γ_5
γ_2	γ_1	γ_2	γ_5	γ_5	γ_5	γ_5
γ_3	γ_0	γ_1	γ_2	γ_5	γ_5	γ_5
γ_4	γ_0	γ_1	γ_2	γ_4	γ_5	γ_5
γ_5	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5

Gödel implication is obtained for the minimum t-norm, and the Łukasiewicz implication is obtained for the Archimedean t-norm.

Definition 10. *A* QL-implication *is an implication verifying* $\gamma_i \Rightarrow \gamma_j = (\ominus\gamma_i) \oplus (\gamma_i \otimes \gamma_j)$.

The following result shows that (in the smooth case) QL-implications only depend on a t-norm. In the non smooth case, this is not true.

Proposition 5. *Let* $\otimes : \mathcal{N}^2 \to \mathcal{N}$ *be a smooth t-norm. The operator* $\gamma_i \Rightarrow_{ql\otimes} \gamma_j = (\ominus\gamma_i) \oplus (\gamma_i \otimes \gamma_j)$ *is a QL-implication iff* $\oplus$ *is the Archimedean smooth t-conorm. Moreover, in this case,* $\gamma_i \Rightarrow_{ql\otimes} \gamma_j = \gamma_{p-i+z}$ *for all* $\gamma_i, \gamma_j \in \mathcal{N}$*, where* $\gamma_z = \gamma_i \otimes \gamma_j$.

Proposition 6. *Let* $\otimes^J : \mathcal{N} \times^J \mathcal{N} \to \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$*. Then, the implication* $\Rightarrow_{ql\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{ql\otimes} \gamma_j = \begin{cases} \gamma_{\max\{p-i+i_k, p+j-i_{k+1}\}} & \textit{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \textit{ for some } k \in [0, m-1] \\ \gamma_{p-i+j} & \textit{if } \gamma_j \leq i_k \leq \gamma_i \textit{ for some } i_k \in J \\ \gamma_p & \textit{otherwise} . \end{cases}$$

The Łukasiewicz implication corresponds to the minimum t-norm, and the Kleene-Dienes implication corresponds to the Archimedean t-norm (note the difference with respect to S-implications).

Interestingly, $\Rightarrow_{s\otimes}$ and $\Rightarrow_{ql\otimes}$ are smooth if and only if so is $\otimes$, but the smoothness condition is not preserved in general for R-implications.

Another interesting operators are D-implications (also called NQL-implications), which generalize the Dishkant arrow in orthomodular lattices.

Definition 11. *A* D-implication *is an implication satisfying* $\gamma_i \Rightarrow \gamma_j = ((\ominus\gamma_i) \otimes (\ominus\gamma_j)) \oplus \gamma_j$ *for all* $\gamma_i, \gamma_j \in \mathcal{N}$.

However, if $\otimes$ is a smooth t-norm, then QL-implications and D-implications on $\mathcal{N}$ actually coincide. Given a set J and $J' = \{\gamma_{p-x} \mid \gamma_x \in J\}$, then $\Rightarrow_{ql\otimes^J}$ is equivalent to $\Rightarrow_{d\otimes^{J'}}$.

In the non-smooth case, a full characterization of the operators is still unknown, and only some partial results are available. However, this a interesting case as it includes popular operators such as the nilpotent minimum (Example 4).

Example 4. The nilpotent minimum is defined as $\gamma_x \otimes \gamma_y = \gamma_0$ if $x + y \leq p$, or $\min\{\gamma_x, \gamma_y\}$ otherwise. For $\mathcal{N} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ we have:

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_0	γ_0	γ_0	γ_0	γ_1
γ_2	γ_0	γ_0	γ_0	γ_0	γ_2	γ_2
γ_3	γ_0	γ_0	γ_0	γ_3	γ_3	γ_3
γ_4	γ_0	γ_0	γ_2	γ_3	γ_4	γ_4
γ_5	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5

The notions of fuzzy relation, inverse relation, composition of relations, reflexivity, symmetry and transitivity can trivially be restricted to $\mathcal{N}$.

2.2 The Description Logic $\mathcal{ALCH}$

Each DL is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. For instance, the language OWL 2 [2], the current W3C recommendation, is close equivalent to $\mathcal{SROIQ}(\mathbf{D})$ [27]. In this section we will quickly recap the main features of the DL $\mathcal{ALCH}$. For more details we refer the reader to [1].

Syntax. $\mathcal{ALCH}$ assumes three alphabets of symbols, for *concepts*, *roles* and *individuals*. In DLs, complex concepts and roles can be built using different concept and role constructors. A Knowledge Base (KB) comprises the intensional knowledge, i.e. axioms about the application domain (a Terminological Box or *TBox* $\mathcal{T}$ and a Role Box or *RBox* $\mathcal{R}$), and the extensional knowledge, i.e. particular knowledge about some specific situation (an Assertional Box or *ABox* $\mathcal{A}$ with axioms about individuals).

The syntax of concept, roles, and axioms of $\mathcal{ALCH}$ is shown in Table 2, where C, D are (possibly complex) concepts, A is an atomic concept, R is a role, and a, b are individuals.

Table 2. Syntax and semantics of the DL $\mathcal{ALCH}$

Element	Name	Syntax	Semantics
Concepts	Atomic concept	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
	Top concept	$\top$	$\Delta^{\mathcal{I}}$
	Bottom concept	$\bot$	$\emptyset$
	Concept conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
	Concept disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
	Concept negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
	Universal quantification	$\forall R.C$	$\{x \mid \forall y, (x, y) \notin R^{\mathcal{I}}$ or $y \in C^{\mathcal{I}}\}$
	Existential quantification	$\exists R.C$	$\{x \mid \exists y, (x, y) \in R^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$
Roles	Atomic role	R	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \otimes \Delta^{\mathcal{I}}$
ABox axioms	Concept assertion	$a : C$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
	Role assertion	$(a, b) : R$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
TBox axioms	GCI	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
RBox axioms	RIA	$R_1 \sqsubseteq R_2$	$R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$

In the KB, concept assertions represent that an individual a is an instance of a concept C, role assertions encode that (a, b) is an instance of R, a *general concept inclusion* (GCI) imposes that C is more specific than D, and a role inclusion axiom (RIA) says that R_1 is more specific than R_2.

Semantics. An interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and an interpretation function $\cdot^{\mathcal{I}}$ mapping:

- every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$,
- every concept C onto a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and
- every role R onto a relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The interpretation is defined as shown in Table 2. A Knowledge Base $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

3 Finite Smooth T-norm Based Fuzzy $\mathcal{ALCH}$

In this section we define fuzzy $\mathcal{ALCH}$, a fuzzy extension of $\mathcal{ALCH}$ where:

- concepts denote fuzzy sets of individuals;
- roles denote fuzzy binary relations;
- degrees of truth are taking from a finite chain $\mathcal{N}$;
- axioms have a degree of truth associated;
- the fuzzy connectives used are a smooth t-norm $\otimes$ on $\mathcal{N}$, the strong negation $\ominus$ on $\mathcal{N}$, the dual t-conorm $\oplus_{\otimes}$, and the implications $\Rightarrow_{s\otimes}, \Rightarrow_{r\otimes}, \Rightarrow_{ql\otimes}$.

3.1 Definition of the Logic

Notation. In the rest of this paper, C, D are (possibly complex) concepts, A is an atomic concept, R is a role, a, b are individuals, $\bowtie \in \{\geq, <, \leq, >\}$, $\lhd \in \{\geq, >\}$, $\rhd \in \{\leq, <\}, \alpha \in \mathcal{N}^+, \beta \in \mathcal{N} \setminus \{\gamma_p\}$. We will also use $\equiv$ to denote semantical equivalence, and we will not write $\otimes$ in the subscripts of the implications.

Syntax. Finite fuzzy $\mathcal{ALCH}$ assumes three alphabets of symbols, for concepts, roles and individuals. A *Fuzzy Knowledge Base* (KB) contains a finite set of axioms organized in a fuzzy ABox $\mathcal{A}$ (axioms about individuals), a fuzzy TBox $\mathcal{T}$ (axioms about concepts), and a fuzzy RBox $\mathcal{T}$ (axioms about roles).

The syntax of fuzzy concept, roles, and axioms is shown in Table 3. We will only allow axioms of the forms $\langle \tau \geq \alpha \rangle$, $\langle \tau > \beta \rangle$, $\langle \tau \leq \beta \rangle$, and $\langle \tau < \alpha \rangle$.

Example 5. The fact that it is likely true that Paul can be considered tall can be encoded using the axiom $\langle \texttt{paul}\colon \texttt{Tall} \geq \texttt{closeToTrue} \rangle$ without needing an explicit numerical degree.

Remark 1. As opposed to the crisp case, there are three types of universal restrictions, fuzzy GCIs, and fuzzy RIAs. In fact, the different subscripts $_s$, $_r$, and $_{ql}$ denote an S-implication, R-implication, and QL-implication, respectively.

Table 3. Syntax and semantics of finite fuzzy $\mathcal{ALCH}$

Element	**Syntax**	**Semantics**
Concepts	$\top$	γ_p
	$\bot$	γ_0
	A	$A^{\mathcal{I}}(x)$
	$C \sqcap D$	$C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
	$C \sqcup D$	$C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$
	$\neg C$	$\ominus C^{\mathcal{I}}(x)$
	$\forall_s R.C$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow_s C^{\mathcal{I}}(y)\}$
	$\forall_r R.C$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow_r C^{\mathcal{I}}(y)\}$
	$\forall_{ql} R.C$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow_{ql} C^{\mathcal{I}}(y)\}$
	$\exists R.C$	$\sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
Roles	R	$R^{\mathcal{I}}(x,y)$
ABox axioms	$\langle a : C \bowtie \gamma \rangle$	$C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$
	$\langle (a,b) : R \bowtie \gamma \rangle$	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \gamma$
TBox axioms	$\langle C \sqsubseteq_s D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow_s D^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle C \sqsubseteq_r D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow_r D^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle C \sqsubseteq_{ql} D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow_{ql} D^{\mathcal{I}}(x)\} \rhd \gamma$
RBox axioms	$\langle R_1 \sqsubseteq_s R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}} \{R_1^{\mathcal{I}}(x) \Rightarrow_s R_2^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle R_1 \sqsubseteq_r R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}} \{R_1^{\mathcal{I}}(x) \Rightarrow_r R_2^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle R_1 \sqsubseteq_{ql} R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}} \{R_1^{\mathcal{I}}(x) \Rightarrow_{ql} R_2^{\mathcal{I}}(x)\} \rhd \gamma$

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non empty set (the interpretation domain) and $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function mapping

- every individual a to an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$,
- every concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to \mathcal{N}$, and
- every role R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathcal{N}$.

The fuzzy interpretation function is extended to fuzzy *complex concepts* and *axioms* as shown in Table 3. $C^{\mathcal{I}}$ denotes the membership function of the fuzzy concept C with respect to the fuzzy interpretation $\mathcal{I}$. $C^{\mathcal{I}}(x)$ gives us the degree of being x an element of the fuzzy concept C under $\mathcal{I}$. Similarly, $R^{\mathcal{I}}$ denotes the membership function of the fuzzy role R with respect to $\mathcal{I}$. $R^{\mathcal{I}}(x,y)$ gives us the degree of being (x,y) an element of the fuzzy role R.

Remark 2. Note an important difference with previous work in fuzzy DLs. Usually, $\cdot^{\mathcal{I}}$ maps every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, and every role R onto $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. Consequently, a fuzzy KB $\{\langle a : C > 0.5\rangle, \langle a : C < 0.75\}$ is satisfiable, by taking $C^{\mathcal{I}}(a) \in (0.5, 0.75)$. But now, given $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$, a fuzzy KB $\{\langle a : C > \texttt{closeToFalse}\rangle, \langle a : C < \texttt{neutral}\}$ is unsatisfiable, since $C^{\mathcal{I}}(a) \in \mathcal{N}$.

Now we will briefly present an example where our logic has a behaviour which is different from both Zadeh, Gödel and Łukasiewicz fuzzy DLs. In some applications, this property could be more appropriate.

Example 6. Consider the finite chain in Example 1 and the pair of axioms $\langle a : C_1 \geq \gamma_2 \rangle$ and $\langle a : C_2 \geq \gamma_2 \rangle$. If we consider Gödel t-norm, $\langle a : C_1 \sqcap C_2 \geq \gamma_2 \rangle$. If we consider Łukasiewicz finite t-norm, $\langle a : C_1 \sqcap C_2 \geq \gamma_0 \rangle$. However, if we consider the t-norm in Example 2, we get an intermediate value: $\langle a : C_1 \sqcap C_2 \geq \gamma_1 \rangle$.

3.2 Logical Properties

It can be easily shown that finite fuzzy $\mathcal{ALCH}$ is a sound extension [28] of crisp $\mathcal{ALCH}$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{\gamma_0 = 0, \gamma_p = 1\}$.

Proposition 7. *Finite fuzzy $\mathcal{ALCH}$ interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{\gamma_0 = 0, \gamma_p = 1\}$.*

The following properties are extensions to a finite chain $\mathcal{N}$ of properties for Zadeh fuzzy DLs [14] and Łukasiewicz fuzzy DLs [16].

1. *Concept simplification*: $C \sqcap \top \equiv C$, $C \sqcup \bot \equiv C$,$C \sqcap \bot \equiv \bot$, $C \sqcup \top \equiv \top$, $\exists R.\bot \equiv \bot$, $\forall_s R.\top \equiv \top$, $\forall_r R.\top \equiv \top$, $\forall_{ql} R.\top \equiv \top$.
2. *Involutive* negation: $\neg\neg C \equiv C$.
3. *Excluded middle and contradiction*: In general, $C \sqcup \neg C \not\equiv \top$, $C \sqcap \neg C \not\equiv \bot$.
4. *Idempotence* of conjunction/disjunction: In general, $C \sqcap C \not\equiv C$, $C \sqcup C \not\equiv C$.
5. *De Morgan* laws: $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$, $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$.
6. *Inter-definability of concepts*: $\bot \equiv \neg\top$, $\top \equiv \neg\bot$, $C \sqcap D \equiv \neg(\neg C \sqcap \neg D)$, $C \sqcup D \equiv \neg(\neg C \sqcup \neg D)$, $\forall_s R.C \equiv \neg\exists R.(\neg C)$, $\exists R.C \equiv \neg\forall_s R.(\neg C)$. However, in general, $C \sqcap D \not\equiv \neg(\neg C \sqcup \neg D)$, $C \sqcup D \not\equiv \neg(\neg C \sqcap \neg D)$, $\forall_r R.C \not\equiv \neg\exists R.(\neg C)$, $\exists R.C \not\equiv \neg\forall_r R.(\neg C)$, $\forall_{ql} R.C \not\equiv \neg\exists R.(\neg C)$, $\exists R.C \not\equiv \neg\forall_{ql} R.(\neg C)$.
7. *Inter-definability of axioms*: $\langle a : C \leq \alpha \rangle \equiv \langle a : \neg C \geq \ominus\alpha \rangle$, $\langle \tau > \beta \rangle \equiv \langle \tau > +\beta \rangle$, $\langle \tau < \alpha \rangle \equiv \langle \tau \leq -\alpha \rangle$.
8. *Contrapositive symmetry*: $C \sqsubseteq_s D \equiv \neg D \sqsubseteq_s \neg C$. However, in general, $C \sqsubseteq_r D \not\equiv \neg D \sqsubseteq_r \neg_s C$,$C \sqsubseteq_{ql} D \not\equiv \neg D \sqsubseteq_{ql} \neg_s C$.
9. *Modus ponens*: $\langle a : C \rhd \gamma_1 \rangle$ and $\langle C \sqsubseteq_r D \rhd \gamma_2 \rangle$ imply $\langle a : D \rhd \gamma_1 \otimes \gamma_2 \rangle$, $\langle (a,b) : R \rhd \gamma_1 \rangle$ and $\langle R \sqsubseteq_r R' \rhd \gamma_2 \rangle$ imply $\langle (a,b) : R' \rhd \gamma_1 \otimes \gamma_2 \rangle$.
10. *Self-subsumption*: $(C \sqsubseteq_r C)^{\mathcal{I}} = \gamma_p$, $(R \sqsubseteq_r R)^{\mathcal{I}} = \gamma_p$. However, in general, $(C \sqsubseteq_s C)^{\mathcal{I}} \neq \gamma_p$, $(R \sqsubseteq_s R)^{\mathcal{I}} \neq \gamma_p$, and $(C \sqsubseteq_{ql} C)^{\mathcal{I}} \neq \gamma_p$, $(R \sqsubseteq_{ql} R)^{\mathcal{I}} \neq \gamma_p$.

Remark 3. Property 7 makes it possible to restrict to fuzzy axioms $\langle \tau \geq \alpha \rangle$ and $\langle \tau \leq \beta \rangle$, as we will do in the rest of this paper.

A fuzzy interpretation $\mathcal{I}$ is *witnessed* iff, for every formula, the infimum corresponds to the minimum and the supremum corresponds to the maximum [12]. Finite fuzzy $\mathcal{ALCH}$ enjoys the Witnessed Model Property (WMP) (all interpretations are witnessed), because the number of degrees of truth in the fuzzy interpretations of the logic is finite [12].

3.3 Reasoning Tasks

Now we will define the most important reasoning tasks and show that all of them can be reduced to fuzzy KB satisfiability. We will use c to denote a new individual, which do not appear in a fuzzy KB $\mathcal{K}$.

- *Fuzzy KB satisfiability.* A fuzzy interpretation $\mathcal{I}$ *satisfies* (is a model of) a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.
- *Concept satisfiability.* C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle c : C \geq \alpha \rangle\}$ is satisfiable.
- *Entailment.* A fuzzy concept assertion $\langle a : C \geq \alpha \rangle$is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \geq \alpha \rangle$) iff $\mathcal{K} \cup \{\langle a : C < \alpha \rangle\}$ is unsatisfiable. Furthermore, $\mathcal{K} \models \langle (a, b) : R \geq \alpha \rangle$ iff $\mathcal{K} \cup \{\langle b : B \geq \gamma_p \rangle\} \models \langle a : \exists R.B \geq \alpha \rangle$ where B is a new concept.
- *Greatest lower bound.* The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle \tau \geq \alpha \rangle\}$. It can be computed performing at most $\log |\mathcal{N}|$ entailment tests [29].
- *Concept subsumption*: There are 3 cases depending on the fuzzy implication:
  - Under an S-implication, D α-subsumes C ($C \sqsubseteq_s D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{c : \neg C \sqcup D < \alpha\}$ is unsatisfiable.
  - Under an R-implication, D subsumes C ($C \sqsubseteq_r D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{c : C \geq \gamma_1\} \cup \{c : D < \gamma_2\}$ is unsatisfiable, for every $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 \otimes \alpha = \alpha_2$ [30].
  - Under a QL-implication, D α-subsumes C ($C \sqsubseteq_{ql} D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{c : \neg C \sqcup (C \sqcap D) < \alpha\}$ is unsatisfiable.

3.4 A Crisp Representation for Finite Fuzzy $\mathcal{ALCH}$

In this section we show how to reduce a fuzzy KB into a crisp KB. The procedure is satisfiability-preserving, so existing DL reasoners could be applied to the resulting KB. The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations. Next, every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

Before proceeding formally, we will illustrate this idea with an example.

Example 7. Consider the smooth t-norm on $\mathcal{N}$ used in Example 2, and let us compute some α-cuts of the fuzzy concept $A_1 \sqcap A_2$ (denoted $\rho(A_1 \sqcap A_2, \geq \alpha)$).

To begin with, let us consider $\alpha = \gamma_2$. By definition, this set includes the elements of the domain x satisfying $A_1^{\mathcal{I}}(x) \otimes A_2^{\mathcal{I}}(x) \geq \gamma_2$. There are two possibilities: *(i)* $A_1^{\mathcal{I}}(x) \geq \gamma_2$ and $A_2^{\mathcal{I}}(x) \geq \gamma_3$, or *(ii)* $A_1^{\mathcal{I}}(x) \geq \gamma_3$ and $A_2^{\mathcal{I}}(x) \geq \gamma_2$. Hence, $\rho(A_1 \sqcap A_2, \geq \gamma_2) = \Big(\rho(A_1, \geq \gamma_2) \sqcap \rho(A_2, \geq \gamma_3)\Big) \sqcup \Big(\rho(A_1, \geq \gamma_3) \sqcap \rho(A_2, \geq \gamma_2)\Big)$.

Now, let us consider $\alpha = \gamma_3$. Now, there is only one possibility: $A_1^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma_3$ and $A_2^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma_3$. Hence, $\rho(A_1 \sqcap A_2, \geq \gamma_3) = \rho(A_1, \geq \gamma_3) \sqcap \rho(A_2, \geq \gamma_3)$.

Observe that for idempotent degrees ($\alpha \in J$) the case is the same as in finite Zadeh and Gödel fuzzy logics [14,15], whereas for non-idempotent degrees it is similar to the case of finite Łukasiewicz fuzzy logic [16].

Adding New Elements. Let $\mathbf{A}$ be the set of atomic fuzzy concepts and $\mathbf{R}$ the set of atomic fuzzy roles in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$, respectively. For each $\alpha \in \mathcal{N}^+$, for each $A \in \mathbf{A}$, a new atomic concepts $A_{\geq \alpha}$ is introduced. $A_{\geq \alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. Similarly, for each $R \in \mathbf{R}$, a new atomic role $R_{\geq \alpha}$ is created.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq p-1$ and for each $A \in \mathbf{A}$, $T(\mathcal{N})$ is the smallest terminology containing the axioms $A_{\geq \gamma_{i+1}} \sqsubseteq A_{\geq \gamma_i}$. Similarly, for each $R_A \in \mathbf{R}$, $R(\mathcal{N})$ is the smallest terminology containing the axioms $R_{\geq \gamma_{i+1}} \sqsubseteq R_{\geq \gamma_i}$.

Remark 4. The atomic elements $A_{\geq \gamma_0}$ and $R_{\geq \gamma_0}$ are not considered because they are always equivalent to the $\top$ concept. Also, as opposite to previous works [14,15,16] we are not introducing elements of the forms $A_{> \beta}$ and $R_{> \beta}$ (for each $\beta \in \mathcal{N} \setminus \{\gamma_p\}$), since now $A_{> \gamma_i}$ is equivalent to $A_{\geq \gamma_{i+1}}$, and $R_{> \gamma_i}$ is equivalent to $R_{\geq \gamma_{i+1}}$. Hence, the number of new axioms needed here is smaller, since we do not need to deal with elements of the forms $A_{> \beta}$ and $R_{> \beta}$.

Mapping Fuzzy Concepts, Roles and Axioms. Fuzzy concept and roles are reduced using mapping ρ as shown in Table 4. Given a fuzzy concept C, $\rho(C, \geq \alpha)$ is a crisp set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C, \bowtie \gamma)$ are similar. ρ is defined in a similar way for fuzzy roles. Furthermore, axioms are reduced as in the bottom part of Table 5, where $\kappa(\tau)$ maps a fuzzy axiom τ in finite fuzzy $\mathcal{ALCH}$ into a set of crisp axioms in $\mathcal{ALCH}$.

The reduction of the conjunction considers every pair $\gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$ such that $\alpha \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\alpha = \gamma_z$. Note that the reduction does not consider a closed interval of the form $[\gamma_{i_k}, \gamma_{i_{k+1}}]$. The reason is that, if α is idempotent and we set $\gamma_{i_{k+1}} = \alpha$, the result is correct ($\gamma_x = \gamma_y = \alpha$). However, setting $\gamma_{i_k} = \alpha$ would yield an incorrect result.

Similarly, the reduction of the disjunction considers every pair $\gamma_1, \gamma_2 \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$ such that $\alpha \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $\gamma_1 + \gamma_2 = \gamma_{i_k} + \alpha$, instead of a closed interval of the form $[\gamma_{i_k}, \gamma_{i_{k+1}}]$.

The crisp representations of R-implications and QL-implications only consider *optimal pairs* of elements, because we want to have efficient representations that avoid the inclusion of superfluous elements. To this end, we need to formally define the notions of optimality of pairs of degrees of truth.

Definition 12. *Let $\odot \in \{\otimes, \oplus\}$ be a fuzzy operator defined in $\mathcal{N}$, and $\gamma_x, \gamma_y \in \mathcal{N}$. The set of* $(\odot_{\geq \alpha})$-optimal pairs *is composed by every pair (γ_x, γ_y) such that:*

- $\gamma_x \odot \gamma_y \geq \alpha$, *and*
- $\nexists \gamma'_x \in \mathcal{N}$ *such that* $\gamma'_x \odot \gamma_y \geq \alpha$ *and* $\gamma'_x < \gamma_x$, *and*
- $\nexists \gamma'_y \in \mathcal{N}$ *such that* $\gamma_x \odot \gamma'_y \geq \alpha$ *and* $\gamma'_y < \gamma_y$.

Let us explain the intuition behind Definition 12 for a t-norm $\otimes$. Assume that there are $\gamma_x, \gamma_y \in \mathcal{N}$ such that $\gamma_x \otimes \gamma_y \geq \alpha$. Since t-norms are non-decreasing in both arguments, $\gamma'_x \otimes \gamma_y \geq \alpha$ trivially holds for every $\gamma'_x \in \mathcal{N}, \gamma'_x > \gamma_x$. Hence, we take the minimal degrees γ_x, γ_y that verify the condition.

The reduction of t-norms, t-conorms and S-implications is implicitly using optimal pairs as well. However, in these cases, we are able to give a general expression to compute these optimal pairs.

Table 4. Mapping of concepts and roles

$\rho(\top, \geq \alpha)$ $\rho(\top, \leq \beta)$	$\top$ $\bot$
$\rho(\bot, \geq \alpha)$ $\rho(\bot, \leq \beta)$	$\bot$ $\top$
$\rho(A, \geq \alpha)$ $\rho(A, \leq \beta)$	$A_{\geq \alpha}$ $\neg A_{\geq + \beta}$
$\rho(\neg C, \geq \alpha)$ $\rho(\neg C, \leq \alpha)$	$\rho(C, \leq \ominus \gamma)$ $\rho(C, \geq \ominus \gamma)$
$\rho(C \sqcap D, \geq \alpha)$	$\bigsqcup_{\gamma_x, \gamma_y} \{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$ for every γ_x, γ_y such that $\alpha, \gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\gamma_z = \alpha$
$\rho(C \sqcap D, \leq \beta)$	$\rho(\neg C \sqcup \neg D, \geq \ominus \beta)$
$\rho(C \sqcup D, \geq \alpha)$	$\rho(C, \geq \alpha) \sqcup \rho(D, \geq \alpha) \bigsqcup_{\gamma_x, \gamma_y} \{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$ for every γ_x, γ_y such that $\alpha, \gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_k + z$, with $\gamma_z = \alpha$
$\rho(C \sqcup D, \leq \beta)$	$\rho(\neg C \sqcap \neg D, \geq \ominus \beta)$
$\rho(\exists R.C, \geq \alpha)$	$\bigsqcup_{\gamma_x, \gamma_y} \{\exists \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$ such that $\gamma \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\gamma_z = \alpha$
$\rho(\exists R.C, \leq \beta)$	$\rho(\forall_s R.(\neg C), \geq \ominus \beta)$
$\rho(\forall_s R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x, \gamma_y} \{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}], \alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - i = z - i_{k+1}$, with $\gamma_z = \alpha$
$\rho(\forall_s R.C, \leq \beta)$	$\rho(\exists R.(\neg C), \geq \ominus \beta)$
$\rho(\forall_r R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x, \gamma_y} \{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\ \geq \alpha})$-optimal
$\rho(\forall_r R.C, \leq \beta)$	$\bigsqcup_{\gamma_x, \gamma_y} \{\exists \rho(R, \geq \gamma_x).\rho(C, \leq \gamma_y)\}$ for every $\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}$ such that γ_x, γ_y are $(\Rightarrow_{r\ \leq \beta})$-optimal
$\rho(\forall_{ql} R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x, \gamma_y} \{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \geq \alpha})$-optimal
$\rho(\forall_{ql} R.C, \leq \beta)$	$\bigsqcup_{\gamma_x, \gamma_y} \{\exists \rho(R, \geq \gamma_x).\rho(C, \leq \gamma_y)\}$ for every $\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \leq \beta})$-optimal
$\rho(R, \geq \alpha)$ $\rho(R, \leq \beta)$	$R_{\geq \alpha}$ $\neg R_{\geq + \beta}$

Table 5. Mapping of axioms

$\kappa(\langle a : C \bowtie \gamma \rangle)$	$\{a : \rho(C, \bowtie \gamma)\}$
$\kappa(\langle (a,b) : R \bowtie \gamma \rangle)$	$\{(a,b) : \rho(R, \bowtie \gamma)\}$
$\kappa(\langle C \sqsubseteq_s D \geq \alpha \rangle)$	$\bigcup \{\rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, $\alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - \gamma_i = z - \gamma_{i_{k+1}}$, with $\gamma_z = \alpha$
$\kappa(\langle C \sqsubseteq_r D \geq \alpha \rangle)$	$\bigcup \{\rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\, \geq \alpha})$-optimal
$\kappa(\langle C \sqsubseteq_{ql} D \geq \alpha \rangle)$	$\bigcup \{\forall \rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\, \geq \alpha})$-optimal
$\kappa(\langle R_1 \sqsubseteq_s R_2 \geq \alpha \rangle)$	$\bigcup \{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, $\alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - \gamma_i = z - \gamma_{i_{k+1}}$, with $\gamma_z = \alpha$
$\kappa(\langle R_1 \sqsubseteq_r R_2 \geq \alpha \rangle)$	$\bigcup \{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\, \geq \alpha})$-optimal
$\kappa(\langle R_1 \sqsubseteq_{ql} R_2 \geq \alpha \rangle)$	$\bigcup \{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\, \geq \alpha})$-optimal

Definition 13. *Let $\Rightarrow$ be a fuzzy implication defined in $\mathcal{N}$, $\gamma_x, \gamma_y \in \mathcal{N}$. We define the set $S = \{(\gamma_x, \gamma_y) \mid \gamma_x \Rightarrow \gamma_y \geq \alpha\}$ to contain every pair of individuals whose implication is at least α. Let $X \subseteq \mathcal{N} \times \mathcal{N}$ be a set of pairs of degrees of truth. We define the mappings $R(X)$ and $L(X)$ as follows:*

- $R(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists (\gamma_x, \gamma'_y) \in X$ *such that* $\gamma'_y < \gamma_y\}$,
- $L(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists (\gamma'_x, \gamma_y) \in X$ *such that* $\gamma'_x < \gamma_x\}$.

The set of $(\Rightarrow_{\geq \alpha})$-optimal pairs *is defined as:* $L(R(S))$.

Similarly, we can define the notion of $(\odot_{\leq \beta})$-optimal pairs as follows.

Definition 14. *Let $\Rightarrow$ be a fuzzy implication defined in $\mathcal{N}$, $\gamma_x, \gamma_y \in \mathcal{N}$, $L(X)$ the mapping in Definition 13 and $S' = \{(\gamma_x, \gamma_y) \mid \gamma_x \Rightarrow \gamma_y \leq \beta\}$. Let $X \subseteq \mathcal{N} \times \mathcal{N}$ be a set of pairs of degrees of truth. We define the mapping $R'(X)$ as follows:*

- $R'(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists (\gamma_x, \gamma'_y) \in X$ *such that* $\gamma'_y > \gamma_y\}$,

The set of $(\Rightarrow_{\leq \beta})$-optimal pairs *is defined as:* $L(R'(S'))$.

Example 8. Consider again the R-implication in Example 3. We can see that:

- The $(\Rightarrow_{\geq \gamma_3})$-optimal pairs in $\mathcal{N}^+$ are (γ_3, γ_3), (γ_2, γ_2), and (γ_1, γ_1).
- The $(\Rightarrow_{\leq \gamma_3})$-optimal pairs in $\mathcal{N}$ are (γ_5, γ_3), (γ_3, γ_2), (γ_2, γ_1), and (γ_1, γ_0).

Note that Definition 12 has an important difference with Definitions 13 and 14. In the two latter cases, not every $\gamma'_x < \gamma_x$ prevents γ_x from taking part of an optimal pair. For instance, in Example 8, (γ_3, γ_3) is an $(\Rightarrow_{\geq \gamma_3})$-optimal pair even if $\gamma_2 < \gamma_3$ and $\gamma_2 \otimes \gamma_3 \Rightarrow \gamma_3$. This definition is designed to use the fact that, for instance $\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)$ implies $\forall \rho(R, \geq \gamma'_x).\rho(C, \geq \gamma_y)$ for every $\gamma'_x \in \mathcal{N}, \gamma'_x > \gamma_x$.

Note also that R-implications are, in general, non smooth (see Example 3). Hence, a pair of elements γ_1, γ_y such that $\gamma_x \Rightarrow_r \gamma_y = \alpha$ might not exist, and thus we have to consider an inequality of the form $\gamma_x \Rightarrow_r \gamma_y \geq \alpha$. In smooth t-norms, t-conorms and QL-implications, $=$ and $\geq$ would yield the same result.

$\kappa(\mathcal{A})$ (resp. $\kappa(\mathcal{T})$, $\kappa(\mathcal{R})$) denotes the union of the reductions of every axiom in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$). $\mathsf{crisp}(\mathcal{K})$ denotes the reduction of a fuzzy KB $\mathcal{K}$. A fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced into a KB $\mathsf{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{N}) \cup \kappa(\mathcal{T}), R(\mathcal{N}) \cup \kappa(\mathcal{R}) \rangle$.

Example 9. Let us show a full example of how the reduction works. To this end, consider the smooth t-norm used in Example 2 and the fuzzy KB $\mathcal{K} = \{\langle a : A \sqcap B \geq \gamma_2 \rangle, \langle a : \neg B \geq \gamma_4 \rangle\}$.

The crisp representation $\mathsf{crisp}(\mathcal{K})$ is computed as follows. $T(\mathcal{N})$ is defined as the set containing the following axioms: $A_{\geq \gamma_2} \sqsubseteq A_{\geq \gamma_1}$, $A_{\geq \gamma_3} \sqsubseteq A_{\geq \gamma_2}$, $A_{\geq \gamma_4} \sqsubseteq A_{\geq \gamma_3}$, $A_{\geq \gamma_5} \sqsubseteq A_{\geq \gamma_4}$, $B_{\geq \gamma_2} \sqsubseteq B_{\geq \gamma_1}$, $B_{\geq \gamma_3} \sqsubseteq B_{\geq \gamma_2}$, $B_{\geq \gamma_4} \sqsubseteq B_{\geq \gamma_3}$, $B_{\geq \gamma_5} \sqsubseteq B_{\geq \gamma_4}$.

Now, let us compute $\kappa(\mathcal{A})$. To this end, we have to map every axiom in the fuzzy ontology. The first one is represented as $a : (A_{\geq \gamma_2} \sqcap B_{\geq \gamma_3}) \sqcup (A_{\geq \gamma_3} \sqcap B_{\geq \gamma_2})$, as shown in Example 7. The second axiom is represented as $a : \neg B_{\geq \gamma_2}$.

It is not difficult to check that both $\mathcal{K}$ and $\mathsf{crisp}(\mathcal{K})$ are unsatisfiable.

Properties of the Reduction

Correctness. The following theorem, showing the logic is decidable and that the reduction preserves reasoning, can be shown.

Theorem 1. *The satisfiability problem in finite fuzzy $\mathcal{ALCH}$ is decidable. Furthermore, a finite fuzzy $\mathcal{ALCH}$ fuzzy KB $\mathcal{K}$ is satisfiable iff $\mathsf{crisp}(\mathcal{K})$ is.*

Complexity. In general, the size of $\mathsf{crisp}(\mathcal{K})$ is $\mathcal{O}(|\mathcal{K}| \cdot |\mathcal{N}|^k)$, being k the maximal depth of the concepts appearing in $\mathcal{K}$. In the particular case of finite Zadeh fuzzy logic, the size of $\mathsf{crisp}(\mathcal{K})$ is $\mathcal{O}(|\mathcal{K}| \cdot |\mathcal{N}|)$ [14]. For other fuzzy operators the case is more complex because we cannot infer the exact values of the degrees of truth, so we need to build disjunctions or conjunctions over all possible degrees of truth.

Modularity. The reduction of an ontology can be reused when adding new axioms if they do not introduce new atomic concepts and roles. In this case, it remains to add the reduction of the new axioms. This allows to compute the reduction of the ontology off-line and update $\mathsf{crisp}(\mathcal{K})$ incrementally. The assumption that the basic vocabulary is fully expressed in the ontology is reasonable because ontologies do not usually change once that their development has finished.

4 Extending Finite Fuzzy $\mathcal{ALCH}$

In this section we will discuss how to extend the previous logic in two ways. Firstly, Section 4.1 will consider alternative fuzzy logic operators (in particular, the non smooth case). Then, Section 4.2 will consider alternative DL constructors, with the aim of obtaining more expressive logics than finite fuzzy $\mathcal{ALCH}$.

4.1 The Non-smooth Case

Up to know, we have defined all the fuzzy operators of the logic by starting from a finite chain $\mathcal{N}$ and from a finite smooth t-norm. In this section we will discuss what happens in the case of non-smooth t-norms.

The first observation is that a full characterization of the operators is unknown yet, and there are only some partial results [20,21,22,23].

Another important point is that in the non-smooth case QL-implications depend of both a t-norm and a t-conorm, so the t-conorm of the language should not be restricted to the dual of the t-norm.

Furthermore, D-implications cannot in general be defined by means of QL-implications, so they should be explicitly considered in the language. Hence, we must add the concept $\forall_d R.C$ and the axioms $C_1 \sqsubseteq_d C_2$ and $R_1 \sqsubseteq_d R_2$, with the obvious semantics.

Even if there is not a complete knowledge of these operators, we can provide a reasoning mechanism with them by using the same ideas as in the reduction of R-implications shown in Section 3. In fact, we can consider any pair of degrees that satisfy the semantics of the constructor, and them we can simplify the expression by only taking the optimal pairs.

Let us also denote by $t \in \{s, r, ql, d\}$ the type of the fuzzy implication function used. The reason is that we are going to provide a reduction of $\forall R.C$ concepts, TBox axioms and RBox axioms that can be used for every type of implication.

Now, we are ready to show the procedure to obtain a crisp representation. Table 6 shows the differences in mapping κ, whereas Table 7 shows the differences in mapping ρ. The concept/role constructors and axioms which are not included in these tables are reduced as shown in Section 3.

Table 6. Mapping of axioms in the case of non-smooth t-norms

$\kappa(\langle C \sqsubseteq_t D \geq \alpha \rangle)$	$\bigcup\{\rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$, for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{t\ \geq\alpha})$-optimal
$\kappa(\langle R_1 \sqsubseteq_t R_2 \geq \alpha \rangle)$	$\bigcup\{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$, for every $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{t\ \geq\alpha})$-optimal

Table 7. Mapping of concepts in the case of non-smooth t-norms

$\rho(C \sqcap D, \geq \alpha)$	$\bigsqcup_{\gamma_x, \gamma_y \in \mathcal{N}^+} \{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$, (γ_x, γ_y) $(\otimes_{\geq\alpha})$-optimal
$\rho(C \sqcup D, \geq \alpha)$	$\rho(C, \geq \alpha) \sqcup \rho(D, \geq \alpha) \bigsqcup_{\gamma_x, \gamma_y \in \mathcal{N}^+} \{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$, (γ_x, γ_y) $(\oplus_{\geq\alpha})$-optimal
$\rho(\exists R.C, \geq \alpha)$	$\bigsqcup_{\gamma_x, \gamma_y \in \mathcal{N}^+} \{\exists \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$, (γ_x, γ_y) $(\otimes_{\geq\alpha})$-optimal
$\rho(\forall_t R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x, \gamma_y \in \mathcal{N}^+} \{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$, (γ_x, γ_y) $(\Rightarrow_{\geq\alpha})$-optimal
$\rho(\forall_t R.C, \leq \beta)$	$\bigsqcup_{\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}} \{\exists \rho(R, \geq \gamma_x).\rho(C, \leq \gamma_y)\}$, (γ_x, γ_y) $(\Rightarrow_{\leq\alpha})$-optimal

Example 10. Let us compare the reductions produced using the t-norms in Example 2 and Example 4 by looking at how they behave when reducing the axiom $\langle a : A \sqcap B \geq \gamma_2 \rangle$.

- On the one hand, using the t-norm in Example 2, we take $\gamma_x, \gamma_y \in (\gamma_0, \gamma_3]$ such that $x + y = 3 + 2 = 5$. Hence, we obtain:

$$a : (A_{\geq \gamma_2} \sqcap B_{\geq \gamma_3}) \sqcup (A_{\geq \gamma_3} \sqcap B_{\geq \gamma_2}) \ .$$

  We recall the reader that this result was obtained from a more intuitive point of view in Example 7.
- On the other hand, using the t-norm in Example 4 the result is different. Now, have to consider the $(\otimes_{\geq \gamma_2})$-optimal pairs, and thus we have:

$$a : (A_{\geq \gamma_4} \sqcap B_{\geq \gamma_2}) \sqcup (A_{\geq \gamma_3} \sqcap B_{\geq \gamma_3}) \sqcup (A_{\geq \gamma_2} \sqcap B_{\geq \gamma_4}) \ .$$

  Note that we have to consider $A_{\geq \gamma_3} \sqcap B_{\geq \gamma_3}$ even if $\gamma_3 \otimes \gamma_3 = \gamma_3 \neq \gamma_2$.

4.2 Other DL Constructors

Our reduction procedure is modular and it could be applied to more expressive DLs. In particular, adding some elements such that their semantics do not depend on any particular choice of fuzzy operators is straightforward because they can be dealt with in the same way as for the Zadeh family [14].

Let S denote a simple fuzzy role[2]. Firstly, we will consider two new concept constructors (fuzzy nominals and local reflexivity concepts) and one new role constructor (inverse roles). The syntax and semantics of these constructors are:

Syntax	Semantics
$\{\alpha/a\}$	α if $x = a^{\mathcal{I}}$, 0 otherwise
$\exists S.\texttt{Self}$	$S^{\mathcal{I}}(x, x)$
R^-	$R^{\mathcal{I}}(y, x)$

Now, we will introduce some new axioms: disjoint, reflexive, irreflexive, symmetric, and asymmetric role axioms. The syntax and semantics of the axioms is defined as follows:

Syntax	Semantics
$\texttt{dis}(S_1, \ldots, S_m)$	$\forall x, y \in \Delta^{\mathcal{I}}, \min\{S_i^{\mathcal{I}}(x, y), S_j^{\mathcal{I}}(x, y)\} = 0, \forall 1 \leq i < j \leq m$
$\texttt{ref}(R)$	$\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, x) = 1$
$\texttt{irr}(S)$	$\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x, x) = 0$
$\texttt{sym}(R)$	$\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, y) = R^{\mathcal{I}}(y, x)$
$\texttt{asy}(S)$	$\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x, y) > 0$ then $S^{\mathcal{I}}(y, x) = 0$

[2] Intuitively, simple roles are such that they do not take part in cyclic role inclusion axioms (see [14] for a formal definition in the context of fuzzy DLs). Simple roles are needed in some parts of a fuzzy KB to guarantee the decidability of the logic.

Mapping ρ can be extended in order to deal with these new concept and role constructors in the following way:

$\rho(\{\gamma/a\}, \geq \alpha)$	$\{a\}$ if $\gamma \geq \alpha$, $\perp$ otherwise
$\rho(\{\gamma/a\}, \leq \beta)$	$\neg\{a\}$ if $\gamma > \beta$, $\top$ otherwise
$\rho(\exists S.\texttt{Self}, \geq \alpha)$	$\exists \rho(S, \geq \alpha).\texttt{Self}$
$\rho(\exists S.\texttt{Self}, \leq \beta)$	$\neg\exists \rho(S, \neg \geq +\beta).\texttt{Self}$
$\rho(R^{-}, \geq \alpha)$	$\rho(R, \geq \alpha)^{-}$
$\rho(R^{-}, \leq \beta)$	$\rho(R, \leq \beta)^{-}$

Furthermore, mapping κ can be extended in order to deal with the new axioms in the following way:

$\kappa(\texttt{dis}(S_1, S_2))$	$\{\texttt{dis}(\rho(S_1, > \gamma_0), \rho(S_2, > \gamma_0))\}$
$\kappa(\texttt{ref}(R))$	$\{\texttt{ref}(\rho(R, \geq \gamma_p))\}$
$\kappa(\texttt{irr}(S))$	$\{\texttt{irr}(\rho(S, > \gamma_0))\}$
$\kappa(\texttt{sym}(R))$	$\bigcup_{\gamma \in \mathcal{N}^{+}} \{\texttt{sym}(\rho(R, \geq \gamma))\}$
$\kappa(\texttt{asy}(S))$	$\{\texttt{asy}(\rho(S, > \gamma_0)\}$

The logic obtained by extending $\mathcal{ALCH}$ with fuzzy nominals and inverse roles is called $\mathcal{ALCHOI}$. Clearly, finite fuzzy $\mathcal{ALCHOI}$ can be mapped into crisp $\mathcal{ALCHOI}$. After having added the other elements, it only remains to represent role inclusion axioms with role chains[3] and qualified cardinality restrictions, in order to cover a fuzzy extension of $\mathcal{SROIQ}$ (and hence OWL 2).

5 Conclusions and Future Work

This paper has set a general framework for fuzzy DLs with a finite chain of degrees of truth $\mathcal{N}$. $\mathcal{N}$ can be seen as a finite totally ordered set of linguistic terms or labels. This is very useful in practice, since expert knowledge is usually expressed using linguistic terms and avoiding their numerical interpretations.

Starting from a smooth finite t-norm on $\mathcal{N}$, we define the syntax and semantics of fuzzy $\mathcal{ALCH}$. The negation function and the t-conorm are imposed by the choice of the t-norm, but there are different options for the implication function. For this reason, whenever this is possible (i.e., in universal restriction concepts and in inclusion axioms), the language allows to use three different implications. We have studied some of the logical properties of the logic. This will help the ontology developers to use the implication that better suit their needs. Hence, our approach makes it possible to use in fuzzy DLs some fuzzy logical operators that have not been considered before.

The decidability of the logic has been shown by presenting a reasoning preserving reduction to the crisp case. Providing a crisp representation for a fuzzy ontology allows reusing current crisp ontology languages and reasoners, among

[3] Most of the works in the DL and fuzzy DL literature also consider transitive role axioms. We have not done so because transitive role axioms can be represented by using role inclusion axioms with role chains.

other related resources. The complexity of the crisp representation is higher than in finite Zadeh fuzzy DLs, because it is necessary to build disjunctions or conjunctions over all possible degrees of truth. However, Zadeh fuzzy DLs have some logical problems [14] which may not be acceptable in some applications, where alternative operators such as those introduced in this paper could be used.

We have also shown how to extend the logic in two directions, by considering non smooth operators, and by considering more expressive DL constructors, obtaining a closer logic to finite fuzzy $\mathcal{SROIQ}$ (and hence finite fuzzy OWL 2).

As future work we would like to study how to reduce qualified cardinality restrictions (see also [16]) and role inclusion axioms with role chains. This way, we will be able to provide the theoretical basis of a general fuzzy extension of OWL 2 under a finite chain of degrees of truth.

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