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Reasoning with the finitely many-valued Łukasiewicz fuzzy Description Logic $\mathcal{SROIQ}$ ☆

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ABSTRACT

Fuzzy Description Logics are a formalism for the representation of structured knowledge affected by imprecision or vagueness. They have become popular as a language for fuzzy ontology representation. To date, most of the work in this direction has focused on the so-called Zadeh family of fuzzy operators (or fuzzy logic), which has several limitations. In this paper, we generalize existing proposals and show how to reason with a fuzzy extension of the logic $\mathcal{SROIQ}$, the logic behind the language OWL 2, under finitely many-valued Łukasiewicz fuzzy logic. We show for the first time that it is decidable over a finite set of truth values by presenting a reasoning preserving procedure to obtain a non-fuzzy representation for the logic. This reduction makes it possible to reuse current representation languages as well as currently available reasoners for ontologies.

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1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning [3].

An ontology is defined as an explicit and formal specification of a shared conceptualization [24], which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies allow adding semantics to data, making knowledge maintenance, information integration, and reuse of components easier.

The current standard language for ontology creation is the Web Ontology Language (OWL [63]), which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL, and OWL Full. OWL Full is the most expressive level, but reasoning within it becomes undecidable; OWL Lite has the lowest complexity; and OWL DL is a balanced tradeoff between expressiveness and reasoning complexity. However, since its first development, several limitations on expressiveness of OWL have been identified, and consequently several extensions to the language have been proposed [55]. Among them, the most significant is OWL 2 [18], its most likely immediate successor which is currently a Proposed Recommendation at W3C [64].

Description Logics (DLs for short) [1] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to

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be very useful as ontology languages [2]. For instance, OWL Lite, OWL DL and OWL 2 are close equivalents to $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SROIQ}(\mathbf{D})$, respectively [31].

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real world domains [49].

Fuzzy logic is a suitable formalism to handle these types of knowledge. In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. A more refined range is used, in such a way that every fuzzy statement has a degree of truth $\alpha \in [0,1]$ [28].

Several fuzzy extensions of DLs can be found in the literature (see [41] for a survey), as the theoretical basis of fuzzy ontologies. Fuzzy ontologies have proved to be useful in several applications, such as Chinese news summarization [39], semantic help-desk support [46], ontology-based query enrichment [37], information retrieval [15], or image interpretation, (e.g. recognition of brain structures in 3D magnetic resonance images [32]). There are also a lot of applications in the Semantic Web field (see for example [47,17]) and, more generally, in the Internet [49].

In fuzzy logic, all classical set operations are extended to the fuzzy case. The intersection, union, complement and implication set operations are performed by a t-norm function $\otimes$, a t-conorm function $\oplus$, a negation function $\ominus$, and an implication function $\Rightarrow$, respectively. These functions or fuzzy operators are grouped in families, also simply called fuzzy logics.

It is well known that different families of fuzzy operators lead to fuzzy DLs with different properties. There are three main fuzzy logics: Łukasiewicz, Gödel and Product. It is also common to consider the fuzzy set operators originally proposed by Zadeh: Gödel conjunction and disjunction, Łukasiewicz negation and Kleene–Dienes implication.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory [41], most of the existing works restrict themselves to Zadeh fuzzy logic (see Section 2.4 for a definition and Section 5 for a detailed summary of the state of the art in fuzzy DLs).

This paper provides a reasoning algorithm for Łukasiewicz fuzzy $\mathcal{SROIQ}$ over a finite set of truth values, the logic behind OWL 2. This is the first reasoning algorithm for such an expressive logic, for which decidability was not known.

Compared to Zadeh logic, our proposal provides several advantages:

- Łukasiewicz fuzzy logic is more general than Zadeh fuzzy logic.
- The implication of Zadeh fuzzy logic (Kleene–Dienes implication) has some counter-intuitive effects [27,4]. For instance, a concept does not fully subsume itself. Łukasiewicz implication solves these problems.
- The t-norm of Zadeh and Gödel fuzzy logics (the minimum t-norm) is idempotent and hence it is not Pareto optimal [48]. This is problematic in some applications such as fuzzy matchmaking [48].

Defining a fuzzy DL brings about that standard languages would no longer be appropriate, new fuzzy languages should be used and hence the large number of resources available should be adapted to the new framework, requiring an important effort. An alternative is to represent fuzzy DLs using non-fuzzy DLs and to reason using these representations. Our reasoning algorithm will provide such a non-fuzzy representation.

The remainder of this work is organized as follows. Section 2 overviews some necessary background. Section 3 describes a fuzzy extension of $\mathcal{SROIQ}$ and particularizes it to the case of Łukasiewicz fuzzy logic. Section 4 depicts a reduction into $\mathcal{SROIQ}$. Section 5 reviews some related work. Finally, Section 6 sets out some conclusions and ideas for future work.

2. Preliminaries

This section provides some basic background. Section 2.1 quickly overviews $\mathcal{SROIQ}$ [30], the DL which will be mainly treated throughout this paper. Then, Section 2.4 refreshes some basic ideas in mathematical fuzzy logic [28].

2.1. The Description Logic $\mathcal{SROIQ}$

$\mathcal{SROIQ}$ extends $\mathcal{ALC}$ standard DL [52] with transitive roles ($\mathcal{ALC}$ plus transitive roles is called $\mathcal{S}$), complex role axioms ($\mathcal{R}$), nominals ($\mathcal{O}$), inverse roles ($\mathcal{I}$) and qualified number restrictions ($\mathcal{Q}$).

2.2. Syntax

$\mathcal{SROIQ}$ assumes three alphabets of symbols, for concepts, roles and individuals. In DLs, complex concepts and roles can be built using different concept and role constructors. In $\mathcal{SROIQ}$, the concepts (denoted C or D) and roles (R) can be built inductively from atomic concepts (A), atomic roles (R_A), top concept $\top$, bottom concept $\bot$, named individuals (o_i), simple roles (S, which will be defined below) and universal role U, as shown in Table 1, where n, m are natural numbers ($n \geqslant 0,\ m > 0$), $x, y \in \Delta^{\mathcal{I}}$ are abstract individuals and $\sharp X$ denotes the cardinality of the set X.

Example 2.1. $\texttt{Man}$ and $\texttt{Woman}$ are atomic concepts. $\texttt{hasChild}$ and $\texttt{likes}$ are atomic roles. $\texttt{Man} \sqcap \geqslant 2\texttt{hasChild}.\texttt{Woman}$ is a complex concept representing a father with at least two daughters. $\exists\texttt{likes}.\texttt{Self}$ represents a narcisist.

Table 1
Syntax and semantics of the Description Logic $\mathcal{SROIQ}$.

Constructor	Syntax	Semantics
(Atomic concept)	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
(Top concept)	$\top$	$\Delta^{\mathcal{I}}$
(Bottom concept)	$\bot$	$\emptyset$
(Concept conjunction)	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
(Concept disjunction)	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
(Concept negation)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
(Universal quantification)	$\forall R.C$	$\{x \vert \forall y, (x,y) \notin R^{\mathcal{I}} \text{ or } y \in C^{\mathcal{I}}\}$
(Existential quantification)	$\exists R.C$	$\{x \vert \exists y, (x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$
(Nominals)	$\{o_1,\ldots,o_m\}$	$\{o_1^{\mathcal{I}}, \ldots, o_m^{\mathcal{I}}\}$
(At-least number restriction)	$\geqslant n\ S.C$	$\{x \vert \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \geqslant n\}$
(At-most number restriction)	$\leqslant n\ S.C$	$\{x \vert \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \leqslant n\}$
(Local reflexivity)	$\exists S.\texttt{Self}$	$\{x \vert (x,x) \in S^{\mathcal{I}}\}$
(Atomic role)	R_A	$R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
(Inverse role)	R^-	$\{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \vert (x,y) \in R^{\mathcal{I}}\}$
(Universal role)	U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

A Knowledge Base (KB) comprises the intensional knowledge, i.e. general knowledge about the application domain (a Terminological Box or *TBox* $\mathcal{T}$ and a Role Box or *RBox* $\mathcal{R}$), and the extensional knowledge, i.e. particular knowledge about some specific situation (an Assertional Box or *ABox* $\mathcal{A}$ with statements about individuals).

An ABox consists of a finite set of *assertions* about individuals:

- *Concept assertions* $a{:}C$, meaning that individual a is an instance of C.
- *Role assertions* $(a,b){:}R$, meaning that (a,b) is an instance of R.
- *Negated role assertions* $(a,b) : \neg R$.
- *Inequality assertions* $a \neq b$.
- *Equality assertions* $a = b$.

A TBox consists of a finite set of *general concept inclusion* (*GCI*) axioms $C \sqsubseteq D$ (C is more specific than D).

Let w be a role chain (a finite string of roles not including the universal role U). An RBox consists of a finite set of role axioms:

- *Role inclusion axioms (RIAs)* $w \sqsubseteq R$ (role chain w is more specific than R).
- *Transitive role axioms* $\texttt{trans}(R)$.
- *Disjoint role axioms* $\texttt{dis}(S_1, S_2)$.
- *Reflexive role axioms* $\texttt{ref}(R)$.
- *Irreflexive role axioms* $\texttt{irr}(S)$.
- *Symmetric role axioms* $\texttt{sym}(R)$.
- *Asymmetric role axioms* $\texttt{asy}(S)$.

Example 2.2. The concept assertion $\texttt{paul}{:}\texttt{Man}$ states that the individual Paul belongs to the class of men. The role assertion $(\texttt{paul}, \texttt{john}) : \neg\texttt{hasChild}$ states that John is not the child of Paul. The GCI $\texttt{Man} \sqsubseteq \texttt{Human}$ states that all men are human. The RIA $\texttt{owns}\ \texttt{hasPart} \sqsubseteq \texttt{owns}$ states the fact if somebody owns something, he also owns its components.

Now we will introduce some definitions which will be useful to impose some limitations in the language. A *strict partial order* $\prec$ on a set A is an irreflexive and transitive relation on A. A strict partial order $\prec$ on the set of roles is called a *regular order* if it also satisfies $R_1 \prec R_2 \iff R_2^- \prec R_1$, for all roles R_1 and R_2.

In order to guarantee the decidability of the logic, there are some restrictions in the use of roles. Given a regular order $\prec$, every role axiom cannot contain U and every RIA should be $\prec$-regular. A RIA $w \sqsubseteq R$ is *$\prec$-regular* if R is atomic and:

1. $w = RR$, or
2. $w = R^-$, or
3. $w = S_1,\ldots,S_n$ and $S_i \prec R$ for all $i = 1,\ldots,n$, or
4. $w = RS_1,\ldots,S_n$ and $S_i \prec R$ for all $i = 1,\ldots,n$, or
5. $w = S_1,\ldots,S_nR$ and $S_i \prec R$ for all $i = 1,\ldots,n$.

Note that, in order to prove decidability of the reasoning, roles are assumed to be simple in some concept constructors (local reflexivity, at-least and at-most number restrictions) and role axioms (disjoint, irreflexive and asymmetric role axioms) [30]. *Simple* roles are defined as follows:

1. R_A is simple if it does not occur on the right side of a RIA.
2. R^- is simple if R is,
3. if R occurs on the right side of a RIA, R is simple if, for each $w \sqsubseteq R$, $w = S$ for a simple role S.

2.3. Semantics

An interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and an interpretation function $\cdot^{\mathcal{I}}$ mapping:

- Every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.
- Every atomic concept A onto a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.
- Every atomic role R_A onto a relation $R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The interpretation is extended to complex concepts and roles by the inductive definitions in Table 1. Unique name assumption is not imposed, i.e. two nominals might refer to the same individual.

Let $\circ$ be the standard composition of relations. An interpretation $\mathcal{I}$ satisfies (is a model of):

- $a{:}C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- $(a, b){:}R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.
- $(a, b) : \neg R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$.
- $a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.
- $a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$.
- $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- $R_1, \ldots, R_n \sqsubseteq R$ iff $R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$.
- $\mathtt{trans}(R)$ iff $(x, y) \in R^{\mathcal{I}}$ and $(y, z) \in R^{\mathcal{I}}$ imply $(x, z) \in R^{\mathcal{I}}, \forall x, y, z \in \Delta^{\mathcal{I}}$.
- $\mathtt{dis}(S_1, S_2)$ iff $S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$.
- $\mathtt{ref}(R)$ iff $(x, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$.
- $\mathtt{irr}(S)$ iff $(x, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$.
- $\mathtt{sym}(R)$ iff $(x, y) \in R^{\mathcal{I}}$ implies $(y, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$.
- $\mathtt{asy}(S)$ iff $(x, y) \in S^{\mathcal{I}}$ implies $(y, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$.
- A Knowledge Base $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}, \mathcal{T}$ and $\mathcal{R}$.

A DL not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability, concept satisfiability or subsumption. However, if a DL is closed under negation, most of the basic reasoning tasks are reducible to KB satisfiability [50], so it is usually the only task considered.

2.4. Mathematical fuzzy logic

In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. A more refined range is used for the function that represents the meaning of a statement. This is usual in natural language when words are modelled by fuzzy sets. For example, the compatibility of "tall" in the phrase "a tall man" with some individual of a given height is often graded: the man can be judged not quite tall, somewhat tall, rather tall, very tall, etc.

Changing the usual true/false convention leads to a new concept of statement, whose compatibility with a given state of facts is a matter of degree and can be measured on e.g., the unit interval $[0,1]$. This degree of fit is called *degree of truth* of the statement ϕ in the interpretation $\mathcal{I}$.

Fuzzy logics provide compositional calculi of degrees of truth, including degrees between "true" and "false". A statement is now not true or false only, but may have a truth degree taken from a *truth space* $\mathcal{S}$, usually $[0,1]$ (in that case we speak about *Mathematical Fuzzy Logic* [28]) or $\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$ for an integer $n \geqslant 1$. Often $\mathcal{S}$ may be also a complete lattice or a bilattice [23,22].

In the illustrative fuzzy logic that we consider in this section, *fuzzy statements* have the form $\phi \geqslant \alpha$ or $\phi \leqslant \beta$, where $\alpha, \beta \in [0,1]$ [26,28] and ϕ is a statement, which encode that the degree of truth of ϕ is *at least l* resp. *at most u*. For example, $\mathsf{ripeTomato} \geqslant 0.9$ says that we have a rather ripe tomato (the degree of truth of $\mathsf{ripeTomato}$ is at least 0.9).

Semantically, a *fuzzy interpretation* $\mathcal{I}$ maps each basic statement p_i into $[0,1]$ and is then extended inductively to all statements as follows:

$$
\begin{aligned}
&\mathcal{I}(\phi \wedge \psi) = \mathcal{I}(\phi) \otimes \mathcal{I}(\psi); \\
&\mathcal{I}(\phi \vee \psi) = \mathcal{I}(\phi) \oplus \mathcal{I}(\psi); \\
&\mathcal{I}(\phi \rightarrow \psi) = \mathcal{I}(\phi) \Rightarrow \mathcal{I}(\psi); \\
&\mathcal{I}(\neg \phi) = \ominus \mathcal{I}(\phi),
\end{aligned} \tag{1}
$$

Table 2
Properties for t-norms and t-conorms.

Axiom name	T-norm	S-norm
Tautology/ contradiction	$\alpha \otimes 0 = 0$	$\alpha \oplus 1 = 1$
Identity	$\alpha \otimes 1 = \alpha$	$\alpha \oplus 0 = \alpha$
Commutativity	$\alpha \otimes \beta = \beta \otimes \alpha$	$\alpha \oplus \beta = \beta \oplus \alpha$
Associativity	$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$	$(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$
Monotonicity	if $\beta \leqslant \gamma$, then $\alpha \otimes \beta \leqslant \alpha \otimes \gamma$	if $\beta \leqslant \gamma$, then $\alpha \oplus \beta \leqslant \alpha \oplus \gamma$

Table 3
Properties for implication and negation functions.

Axiom name	Implication function	Negation function
Tautology/ contradiction	$0 \Rightarrow \beta = 1$, $\alpha \Rightarrow 1 = 1$, $1 \Rightarrow 0 = 0$	$\ominus 0 = 1$, $\ominus 1 = 0$
Antitonicity	if $\alpha \leqslant \beta$, then $\alpha \Rightarrow \gamma \geqslant \beta \Rightarrow \gamma$	if $\alpha \leqslant \beta$, then $\ominus\alpha \geqslant \ominus\beta$
Monotonicity	if $\beta \leqslant \gamma$, then $\alpha \Rightarrow \beta \leqslant \alpha \Rightarrow \gamma$	

where $\otimes$, $\oplus$, $\Rightarrow$, and $\ominus$ are so-called *combination functions*, namely, *triangular norms* (or *t-norms*), *triangular co-norms* (or *t-conorms*), *implication functions*, and *negation functions*,[1] respectively, which extend the classical Boolean conjunction, disjunction, implication, and negation, respectively, to the fuzzy case.

Several t-norms, t-conorms, implication functions, and negation functions have been given in the literature. An important aspect of such functions is that they satisfy some properties that one expects to hold for the connectives; see Tables 2 and 3. Note that in Table 2, the two properties Tautology and Contradiction follow from Identity, Commutativity, and Monotonicity.

Some t-norms, t-conorms, implication functions, and negation functions of various fuzzy logics are shown in Table 4 [28]. In fuzzy logic, one usually distinguishes three different logics, namely, Łukasiewicz (denoted Ł), Gödel (denoted *G*), and Product logic (denoted Π). The importance of these three logics is due the fact that any continuous t-norm can be obtained as a combination of Łukasiewicz, Gödel, and Product t-norm [44].

The usually called "Zadeh logic" is a sublogic of Łukasiewicz fuzzy logic. Some salient properties of these four logics are shown in Table 5. For more properties, see especially [28,45]. Note also, that a fuzzy logic having all properties shown in Table 5, collapses to boolean logic, i.e. the truth-set can be {0,1} only.

An *involutive* negation satisfies that $\ominus(\ominus \alpha) = \alpha$. Łukasiewicz negation is involutive, while Gödel negation is not.

Usually, the implication function $\Rightarrow$ is defined as an *R-implication*, or the residuum of a t-norm $\otimes$, that is, $\alpha \Rightarrow \beta = \sup\{\gamma | \alpha \otimes \gamma \leqslant \beta\}$. An *S-implication* is defined as $\alpha \Rightarrow \beta = \ominus\alpha \oplus \beta$. Łukasiewicz implication is both an R-implication and an S-implication. Gödel and Product logics have an R-implication, whereas Zadeh logic has an S-implication.

The implication $\alpha \Rightarrow \beta = \max(1 - \alpha, \beta)$ is called *Kleene–Dienes implication* in the fuzzy logic literature. Note that we have the following inferences: Let $\phi \geqslant \alpha$ and $\phi \Rightarrow \psi \geqslant \beta$. Then, under Kleene–Dienes implication, we infer that if $\alpha > 1 - \beta$ then $\psi \geqslant \beta$. Under an R-implication relative to a t-norm $\otimes$, we infer that $\psi \geqslant \alpha \otimes \beta$ instead.

Note that implication functions and t-norms are also used to define the degree of subsumption between fuzzy sets and the composition of two (binary) fuzzy relations. A *fuzzy set R* over a countable classical set *X* is a function $R{:}X \to [0,1]$. The *degree of subsumption* between two fuzzy sets *A* and *B*, denoted $A \sqsubseteq B$, is defined as $\inf_{x \in X}\{A(x) \Rightarrow B(x)\}$, where $\Rightarrow$ is an implication function. Note that if $A(x) \leqslant B(x)$, for all $x \in [0,1]$, then $A \sqsubseteq B$ evaluates to 1. Of course, $A \sqsubseteq B$ may evaluate to a value $\alpha \in (0,1)$ as well.

A (binary) *fuzzy relation R* over two countable classical sets *X* and *Y* is a function $R{:}X \times Y \to [0,1]$. The *inverse* of *R* is the function $R^{-1}{:}Y \times X \to [0,1]$ with membership function $R^{-1}(y,x) = R(x,y)$, for every $x \in X$ and $y \in Y$. The *composition* of two fuzzy relations $R_1{:}X \times Y \to [0,1]$ and $R_2{:}Y \times Z \to [0,1]$ is defined as $(R_1 \circ R_2)(x,z) = \sup_{y \in Y} R_1(x,y) \otimes R_2(y,z)$. A fuzzy relation *R* is *transitive* iff $R(x,z) \geqslant (R \circ R)(x,z)$.

A fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy statement $\phi \geqslant l$ (resp., $\phi \leqslant u$) or $\mathcal{I}$ is a *model* of $\phi \geqslant l$ (resp., $\phi \leqslant u$), denoted $\mathcal{I} \models \phi \geqslant l$ (resp., $\mathcal{I} \models \phi \leqslant u$), iff $\mathcal{I}(\phi) \geqslant l$ (resp., $\mathcal{I}(\phi) \leqslant u$). The notions of satisfiability and logical consequence are defined in the standard way. We say that $\phi \geqslant l$ is a *tight logical consequence* of a set of fuzzy statements $\mathcal{K}$ iff *l* is the infimum of $\mathcal{I}(\phi)$ subject to all models $\mathcal{I}$ of $\mathcal{K}$. Notice that the latter is equivalent to $l = \sup\{r | \mathcal{K} \models \phi \geqslant r\}$. We refer the reader to [28] for reasoning algorithms for fuzzy propositional and First-Order Logics.

[1] Note that $\ominus$ is also used in the context of Lukasiewicz logic to denote the binary connective $x \ominus y = max(0;x - y)$. However, in this paper we will use it as a unary negation function.

Table 4
Combination functions of various fuzzy logics.

	Łukasiewicz fuzzy logic	Gödel logic	Product logic	Zadeh logic
$\alpha \otimes \beta$	$\max(\alpha + \beta - 1, 0)$	$\min(\alpha, \beta)$	$\alpha \cdot \beta$	$\min(\alpha, \beta)$
$\alpha \oplus \beta$	$\min(\alpha + \beta, 1)$	$\max(\alpha, \beta)$	$\alpha + \beta - \alpha \cdot \beta$	$\max(\alpha, \beta)$
$\alpha \Rightarrow \beta$	$\min(1 - \alpha + \beta, 1)$	$\begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ \beta & \text{otherwise} \end{cases}$	$\min(1, \beta/\alpha)$	$\max(1 - \alpha, \beta)$
$\ominus\alpha$	$1 - \alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$1 - \alpha$

Table 5
Some additional properties of combination functions of various fuzzy logics.

Property	Łukasiewicz fuzzy logic	Gödel logic	Product logic	Zadeh logic
$\alpha \otimes \ominus\alpha = 0$	+	+	+	–
$\alpha \oplus \ominus\alpha = 1$	+	–	–	–
$\alpha \otimes \alpha = \alpha$	–	+	–	+
$\alpha \oplus \alpha = \alpha$	–	+	–	+
$\ominus\ominus\alpha = \alpha$	+	–	–	+
$\alpha \Rightarrow \beta = \ominus\alpha \oplus \beta$	+	–	–	+
$\ominus(\alpha \Rightarrow \beta) = \alpha \otimes \ominus\beta$	+	–	–	+
$\ominus(\alpha \otimes \beta) = \ominus\alpha \oplus \ominus\beta$	+	+	+	+
$\ominus(\alpha \oplus \beta) = \ominus\alpha \otimes \ominus\beta$	+	+	+	+

3. Fuzzy $\mathcal{SROIQ}$

In this section, we define a fuzzy extension of the DL $\mathcal{SROIQ}$ where concepts denote fuzzy sets of individuals and roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. The following definition is based on the fuzzy DLs presented in [56,4,53].

In the rest of the paper we will assume $\bowtie \in \{\geqslant, <, \leqslant, >\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$ and $\gamma \in [0,1]$. The symmetric $\bowtie^-$ and the negation $\neg\bowtie$ of an operator $\bowtie$ are defined as follows:

$\bowtie$	$\bowtie^-$	$\neg\bowtie$
$\geqslant$	$\leqslant$	$<$
$>$	$<$	$\leqslant$
$\leqslant$	$\geqslant$	$>$
$<$	$>$	$\geqslant$

3.1. Syntax

Fuzzy $\mathcal{SROIQ}$ assumes three alphabets of symbols, for concepts, roles and individuals. Let U be the universal role, and R_A an atomic role. The roles of the language are built using the syntax rule[2]:

$$R \rightarrow R_A | U | R^- \tag{2}$$

The concepts of the language (denoted C or D) can be built inductively from atomic concepts (A), top concept $\top$, bottom concept $\bot$, named individuals (o_i) and roles (R and S, where S is a simple role as defined below) according to the following syntax rule (with n, m being natural numbers, $n \geqslant 0$, $m > 0$):

$$C, D \rightarrow \ A | \top | \bot | C \sqcap D | C \sqcup D | \neg C | \forall R.C | \exists R.C | \{\alpha_1/o_1, \ldots, \alpha_m/o_m\} | (\geqslant mS.C) | (\leqslant nS.C) | \exists S.\mathtt{Self} \tag{3}$$

The only difference with the non-fuzzy case is the presence of fuzzy nominals of the form $\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}$ [4]. We assume that $o_i \neq o_j$, $1 \leqslant i < j \leqslant m$.

Example 3.1. {1/`germany`, 1/`austria`, 0.67/`switzerland`} represents the concept of German-speaking country, with Germany and Austria fully belonging to it, but Switzerland belonging only with degree 0.67, since only about two thirds of its population speak German.

[2] We will also allow role negation in fuzzy assertions of the form $\langle (a, b) : \neg R \bowtie \alpha \rangle$.

A fuzzy KB $\mathcal{K}$ comprises a fuzzy ABox $\mathcal{A}$, a fuzzy TBox $\mathcal{T}$ and a fuzzy RBox $\mathcal{R}$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions*. A fuzzy assertion can be an inequality assertion $\langle a \neq b \rangle$, an equality assertion $\langle a = b \rangle$ or a constraint on the truth value of a concept or role assertion, i.e., an expression of the form $\langle \Psi \geqslant \alpha \rangle$, $\langle \Psi > \beta \rangle$, $\langle \Psi \leqslant \beta \rangle$ or $\langle \Psi < \alpha \rangle$, where Ψ is of the form $a{:}C$, $(a, b){:}R$ or $(a, b) : \neg R$.

A *fuzzy TBox* consists of *fuzzy GCIs*, which constrain the truth value of a GCI i.e. they are expressions of the form $\langle C \sqsubseteq D \geqslant \alpha \rangle$ or $\langle C \sqsubseteq D > \beta \rangle$.

A *fuzzy RBox* consists of a finite set of role axioms, which can be *fuzzy RIAs* $\langle w \sqsubseteq R \geqslant \alpha \rangle$ or $\langle w \sqsubseteq R > \beta \rangle$ for a role chain $w = R_1R_2, \ldots, R_n$, or any other of the role axioms from the non-fuzzy case: *transitive* $\mathtt{trans}(R)$, *disjoint* $\mathtt{dis}(S_1, S_2)$, *reflexive* $\mathtt{ref}(R)$, *irreflexive* $\mathtt{irr}(S)$, *symmetric* $\mathtt{sym}(R)$ or *asymmetric* $\mathtt{asy}(S)$.

Example 3.2. The fuzzy concept assertion $\langle \mathtt{paul{:}Tall} \geqslant 0.5 \rangle$ states that Paul is tall with at least degree 0.5. The fuzzy RIA $\langle \mathtt{isFriendOf\ isFriendOf} \sqsubseteq \mathtt{isFriendOf} \geqslant 0.75 \rangle$ states that the friends of my friends can also be considered as my friends with at least degree 0.75.

A fuzzy axiom is *positive* (denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geqslant \alpha \rangle$ or $\langle \tau > \beta \rangle$, and *negative* (denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leqslant \beta \rangle$ or $\langle \tau < \alpha \rangle$.

$\langle \tau = \alpha \rangle$ is equivalent to the pair of axioms $\langle \tau \geqslant \alpha \rangle$ and $\langle \tau \leqslant \alpha \rangle$ [29]. Of course, $\tau \equiv \langle \tau \geqslant 1 \rangle$.

Notice that negative fuzzy GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs, respectively, which are not part of $\mathcal{SROIQ}$.

As in the non-fuzzy case, role axioms cannot contain U and every RIA should be $\prec$-regular for a regular order $\prec$. A RIA $\langle w \sqsubseteq R \rhd \gamma \rangle$ is *$\prec$-regular* if R is atomic and:

- $w = RR$, or
- $w = R^-$, or
- $w = S_1, \ldots, S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
- $w = RS_1, \ldots, S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
- $w = S_1, \ldots, S_nR$ and $S_i \prec R$ for all $i = 1, \ldots, n$.

Simple roles are defined as in the non-fuzzy case:

- R_A is simple if it does not occur on the right side of a RIA.
- R^- is simple if R is.
- If R occurs on the right side of a RIA, R is simple if, for each $\langle w \sqsubseteq R \rhd \gamma \rangle$, $w = S$ for a simple role S.

3.2. Semantics

A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

- Every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.
- Every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$.
- Every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$.

$C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) denotes the membership function of the fuzzy concept C (resp. fuzzy role R) w.r.t. $\mathcal{I}$. $C^{\mathcal{I}}(a)$ (resp. $R^{\mathcal{I}}(a, b)$) gives us to what extent the individual a can be considered as an element of the fuzzy concept C (resp. to what extent (a, b) can be considered as an element of the fuzzy role R) under the fuzzy interpretation $\mathcal{I}$.

Given a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

$$
\begin{aligned}
&\top^{\mathcal{I}}(x) = 1\\
&\bot^{\mathcal{I}}(x) = 0\\
&(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)\\
&(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)\\
&(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)\\
&(\forall R.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)\}\\
&(\exists R.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)\}\\
&\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}^{\mathcal{I}}(x) = \sup\{\alpha_i \mid x = o_i^{\mathcal{I}}\}
\end{aligned}
$$

$$(\geqslant mS.C)^{\mathcal{I}}(x) = \sup_{y_1,\ldots,y_m \in \Delta^{\mathcal{I}}} (\min_{i=1}^{m}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes \left(\bigotimes_{1\leqslant j<k\leqslant m}\{y_j \neq y_k\}\right)$$
$$(\leqslant nS.C)^{\mathcal{I}}(x) = \inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} (\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow \left(\bigoplus_{1\leqslant j<k\leqslant n+1}\{y_j = y_k\}\right)$$
$$(\exists S.\texttt{Self})^{\mathcal{I}}(x) = S^{\mathcal{I}}(x,x)$$
$$(R^-)^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$$
$$U^{\mathcal{I}}(x,y) = 1$$

We do not impose unique name assumption, i.e. two nominals might refer to the same individual.

The fuzzy interpretation function is extended to fuzzy axioms as follows:

$$(a : C)^{\mathcal{I}} = C^{\mathcal{I}}(a^{\mathcal{I}})$$
$$((a,b) : R)^{\mathcal{I}} = R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$$
$$((a,b) : \neg R)^{\mathcal{I}} = \ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$$
$$(C \sqsubseteq D)^{\mathcal{I}} = \inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\}$$
$$(R_1 \ldots R_n \sqsubseteq R)^{\mathcal{I}} = \inf_{x_1,x_{n+1}\in\Delta^{\mathcal{I}}}\left\{\sup_{x_2\ldots x_n\in\Delta^{\mathcal{I}}}\{(R_1^{\mathcal{I}}(x_1,x_2)\otimes\cdots\otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})\}\right\}$$

Note that this is the semantics for fuzzy RIAs that is implicitly assumed in the non-fuzzy representations provided in [7,5,8].

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- $\langle a{:}C \bowtie \gamma\rangle$ iff $(a : C)^{\mathcal{I}} \bowtie \gamma$.
- $\langle (a,\, b){:}R \bowtie \gamma\rangle$ iff $((a,b) : R)^{\mathcal{I}} \bowtie \gamma$.
- $\langle (a,b) : \neg R \bowtie \gamma\rangle$ iff $((a,b) : \neg R)^{\mathcal{I}} \bowtie \gamma$.
- $\langle a \neq b\rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.
- $\langle a = b\rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$.
- $\langle C \sqsubseteq D \rhd \gamma\rangle$ iff $(C \sqsubseteq D)^{\mathcal{I}} \rhd \gamma$.
- $\langle R_1,\ldots,R_n \sqsubseteq R \rhd \gamma\rangle$ iff $(R_1 \ldots R_n \sqsubseteq R)^{\mathcal{I}} \rhd \gamma$.
- $\texttt{trans}(R)$ iff $\forall x,y,z \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y) \leqslant R^{\mathcal{I}}(x,y)$.
- $\texttt{dis}(S_1,S_2)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = 0$ or $S_2^{\mathcal{I}}(x,y) = 0$.
- $\texttt{ref}(R)$ iff $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$.
- $\texttt{irr}(S)$ iff $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$.
- $\texttt{sym}(R)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$.
- $\texttt{asy}(S)$ iff $\forall x,y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$.
- a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

Notice that individual assertions are either true or false, as it has always been assumed in the fuzzy DL literature [56,54].

Given a fuzzy KB $\mathcal{K}$, the problem of *fuzzy KB satisfiability* consists on checking the existence of a fuzzy interpretation satisfying $\mathcal{K}$. In the rest of the paper we will only consider fuzzy KB satisfiability, since (as in the non-fuzzy case) most inference problems can be reduced to it [60].

Example 3.3. The following tasks can be reduced to fuzzy KB satisfiability:

- *Concept satisfiability*. C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle a : C \geqslant \alpha\rangle\}$ is satisfiable, where a is a new individual, which does not appear in $\mathcal{K}$.
- *Entailment*: A fuzzy concept assertion $a{:}C \bowtie \alpha$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \bowtie \alpha\rangle$) iff $\mathcal{K} \cup \{\langle a : C \neg \bowtie \alpha\rangle\}$ is unsatisfiable. The case for fuzzy role assertions is similar.
- *Greatest lower bound*. The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle\tau \geqslant \alpha\rangle\}$. In Zadeh and finitely many-valued Łukasiewicz and Gödel fuzzy logics, it can be computed performing several entailment tests.[3]
- *Concept subsumption*: Under an S-implication, D subsumes C with degree α ($C \sqsubseteq D \geqslant \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{a : C \sqcap \neg D < \alpha\}$ is unsatisfiable, where a is a new individual.

[3] More precisely, in finitely many-valued Łukasiewicz and Gödel fuzzy logics we need to assume a finite set of degrees of truth $\mathcal{TV}$ including 0 and 1 [27], and the number of tests is at most $log|\mathcal{TV}|$ [60].

Another important notion is that of *witnessed* interpretations. A fuzzy interpretation $\mathcal{I}$ is *witnessed* [27] iff it verifies:

- $\forall x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\exists R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$, and
- $\forall x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\forall R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$, and
- There is $x \in \Delta^{\mathcal{I}}$ such that $(C \sqsubseteq D)^{\mathcal{I}} = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$, and
- There are $x_1, \ldots, x_{n+1} \in \Delta^{\mathcal{I}}$ such that $(R_1 \ldots R_n \sqsubseteq R)^{\mathcal{I}} = (R_1^{\mathcal{I}}(x_1,x_2) \otimes \cdots \otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})$, and
- If $\mathcal{I} \models \texttt{trans}(R)$, for all $x,y \in \Delta^{\mathcal{I}}$, there is $z \in \Delta^{\mathcal{I}}$ such that $\sup_{z' \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z') \otimes R^{\mathcal{I}}(z',y) = R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$.

3.3. Łukasiewicz fuzzy logic

From now on we will concentrate on Ł$\mathcal{SROIQ}$, restricting ourselves to the fuzzy operators of the Łukasiewicz fuzzy logic.

It can be easily shown that Ł$\mathcal{SROIQ}$ is a sound extension of $\mathcal{SROIQ}$, in the sense that fuzzy interpretations coincide with non-fuzzy interpretations if we restrict the degrees of truth to {0,1}.

In Łukasiewicz logic, there are a lot of equivalences which allow the inter-definition of most of the concept constructors. The inter-definability of cardinality restrictions will be specially interesting for us. These equivalences are:

- $\neg(\neg C) \equiv C$.
- $\top \equiv \neg \bot$.
- $\bot \equiv \neg \top$.
- $C \sqcap D \equiv \neg(\neg C \sqcup \neg D)$.
- $C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$.
- $\forall R.C \equiv \neg(\exists R.\neg C)$.
- $\exists R.C \equiv \neg(\forall R.\neg C)$.
- $(\geqslant mS.C) \equiv \neg(\leqslant m-1S.C)$.
- $(\leqslant nS.C) \equiv \neg(\geqslant n+1S.C)$.

The semantics of qualified cardinality restrictions has been proposed in [12] and verifies the following additional properties:

- If $(\leqslant nR.C)^{\mathcal{I}}(a) = 1$ then $|\{b | (R(a,b)^{\mathcal{I}} \otimes C(b))^{\mathcal{I}} > 0\}| \leqslant n$.
- $\exists S.C \equiv \geqslant 1S.C$.

It has been shown that Łukasiewicz fuzzy logic verifies the Witnessed Model Property (WMP), i.e. for each countable model there is an equivalent witnessed model [27]. Hence, we can restrict ourselves to witnessed models.

In Łukasiewicz logic, there are several axioms which are syntactic sugar (and consequently it can be assumed that they do not appear in fuzzy KBs) due to the following equivalences:

Proposition 3.4. *In Ł$\mathcal{SROIQ}$, the following equivalences hold:*

- $\langle (a,b) : \neg R \bowtie \gamma \rangle \equiv \langle (a,b) : R \bowtie^{-} 1-\gamma \rangle$.
- $\texttt{irr}(S) \equiv \langle \top \sqsubseteq \neg \exists S.\texttt{Self} \geqslant 1 \rangle$.
- $\texttt{trans}(R) \equiv \langle R\,R \sqsubseteq R \geqslant 1 \rangle$.
- $\texttt{sym}(R) \equiv \langle R \sqsubseteq R^{-} \geqslant 1 \rangle$.

Finally, the finitely many-valued fragment of Ł$\mathcal{SROIQ}$ is denoted as Ł$_n\mathcal{SROIQ}$, for some natural n.

4. A non-fuzzy representation for fuzzy Ł$_n\mathcal{SROIQ}$

In this section we show how to reduce a Ł$_n\mathcal{SROIQ}$ fuzzy KB into a non-fuzzy KB whenever a finite truth space is assumed. We will start by presenting our reduction procedure, and then we will illustrate it with a couple of examples. Then we will discuss the properties of the reduction, showing that it preserves reasoning, so existing $\mathcal{SROIQ}$ reasoners could be applied to the resulting KB.

The basic idea behind the reduction procedure is to create some new non-fuzzy concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new non-fuzzy elements.

4.1. Adding new elements

It has been shown that, for a fuzzy KB $\mathcal{K}$ under Zadeh logic, the set of the degrees of truth which must be considered for any reasoning task is defined as $N^{\mathcal{K}} = X^{\mathcal{K}} \cup \{1-\alpha | \alpha \in X^{\mathcal{K}}\}$, where $X^{\mathcal{K}} = \{0, 0.5, 1\} \cup \{\gamma | \langle \tau \bowtie \gamma \rangle \in \mathcal{K}\}$ [61]. This holds for fuzzy

DLs under Zadeh fuzzy logic, but it is not true in general when other fuzzy operators are considered. Interestingly, in the case of Łukasiewicz fuzzy logic it is true if we fix the number of allowed degrees.

In fact, let n be a natural number with $n \geqslant 1$. We assume a set of $q+1$ allowed truth degrees in the fuzzy KB, i.e. $\mathcal{N} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\}$. The following proposition shows that, using the fuzzy operators of Łukasiewicz fuzzy logic to combine two truth degrees a and b, no new degrees can appear.

Proposition 4.1. *Let $\frac{a}{n}, \frac{b}{n} \in \mathcal{N}$. Then, under the fuzzy operators of Łukasiewicz fuzzy logic, $\ominus \frac{a}{n}, \frac{a}{n} \otimes \frac{b}{n}, \frac{a}{n} \oplus \frac{b}{n}, \frac{a}{n} \Rightarrow \frac{b}{n} \in \mathcal{N}$ [16].*

This result can be easily checked by considering the four fuzzy operators:

- $\ominus \frac{a}{n} = 1 - \frac{a}{n} = \frac{n-a}{n}$ belongs to $\mathcal{N}$: since $a \in [0,n]$ and $(n-a) \in [0,n]$.
- $\frac{a}{n} \otimes \frac{b}{n} = \max\{\frac{a}{n} + \frac{b}{n} - 1, 0\}$. If $\frac{a}{n} + \frac{b}{n} - 1 \leqslant 0$, then the value of the conjunction is 0, which obviously belongs to $\mathcal{N}$. Otherwise, its value is $\frac{a+b-n}{n}$ which also belongs to $\mathcal{N}$: since $a, b \in [0,n]$ and $\frac{a}{n} + \frac{b}{n} - 1 > 0$, it follows that $(a+b-n) \in [0,n]$.
- $\frac{a}{n} \oplus \frac{b}{n} = \min\{\frac{a}{n} + \frac{b}{n}, 1\}$. If $\frac{a}{n} + \frac{b}{n} > 1$, then the value of the disjunction is 1, which obviously belongs to $\mathcal{N}$. Otherwise, its value is $\frac{a+b}{n}$ which also belongs to $\mathcal{N}$: since $a,b \in [0,n]$ and $\frac{a}{n} + \frac{b}{n} \leqslant 1$, it follows that $(a+b) \in [0,n]$.
- $\frac{a}{n} \Rightarrow \frac{b}{n} = \min\{1 - \frac{a}{n} + \frac{b}{n}, 1\}$. If the minimum is 1, then the value of the implication obviously belongs to $\mathcal{N}$. Otherwise, the value is $\frac{n-a+b}{n}$ which also belongs to $\mathcal{N}$: since $a,b \in [0,n]$ and $1 - \frac{a}{n} + \frac{b}{n} \leqslant 1$, it follows that $(1-a+b) \in [0,n]$.

Note that, given a new individual a, and $n \in \mathcal{N}$, we may always add a fuzzy assertion $\langle a{:}\top \geqslant n\rangle$ to a fuzzy KB without changing its meaning.

Now, we will assume that $N^{\mathcal{K}} = \mathcal{N}$ and proceed similarly as in [7], which creates an optimized number of new elements (concepts, roles and axioms) with respect to previous approaches.

Without loss of generality, it can be assumed that $N^{\mathcal{K}} = \{\gamma_1, \ldots, \gamma_{|N^{\mathcal{K}}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leqslant i \leqslant |N^{\mathcal{K}}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|N^{\mathcal{K}}|} = 1$. We define $\mathcal{N}^+ = \{x \in \mathcal{N} : x \neq 0\}$.

Let $\mathbf{A}$ and $\mathbf{R}$ be the set of fuzzy atomic concepts and fuzzy atomic roles occurring in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$, respectively.

For each $\alpha, \beta \in \mathcal{N}$ with $\alpha \in (0,1]$ and $\beta \in [0,1)$, for each $A \in \mathbf{A}$, two new atomic concepts $A_{\geqslant\alpha}$, $A_{>\beta}$ are introduced. $A_{\geqslant\alpha}$ represents the (non-fuzzy) set of individuals which are instance of A with degree higher or equal than α i.e. the α-cut of A. $A_{>\beta}$ is defined in a similar way.

Similarly, for each $R_A \in \mathbf{R}$ two new atomic roles $R_{A\geqslant\alpha}$, $R_{A>\beta}$. The atomic elements $A_{>1}$, $R_{A>1}$, $A_{\geqslant 0}$ and $R_{A\geqslant 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB (e.g. we do not allow GCIs of the form $C \sqsubseteq D \geqslant 0$).

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leqslant i \leqslant |\mathcal{N}| - 1, 2 \leqslant j \leqslant |\mathcal{N}| - 1$ and for each $A \in \mathbf{A}$, $T(\mathcal{N})$ is the smallest terminology containing these two axioms:

$$\begin{aligned} &A_{\geqslant\gamma_{i+1}} \sqsubseteq A_{>\gamma_i} \\ &A_{>\gamma_j} \sqsubseteq A_{\geqslant\gamma_j} \end{aligned} \tag{4}$$

Similarly, for each $R_A \in \mathbf{R}, R(\mathcal{N})$ is the smallest terminology containing:

$$\begin{aligned} &R_{A\geqslant\gamma_{i+1}} \sqsubseteq R_{A>\gamma_i} \\ &R_{A>\gamma_i} \sqsubseteq R_{A\geqslant\gamma_i} \end{aligned} \tag{5}$$

4.2. Mapping fuzzy concepts, roles and axioms

The reduction of concept and role expressions is achieved using a mapping ρ. Given a fuzzy concept C, $\rho(C, \geqslant\alpha)$ is a set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C, \bowtie\gamma)$ and $\rho(R, \bowtie\gamma)$ for a role R are similar.

In Zadeh logic, due to the definition of the fuzzy operators, it is possible to infer exactly degrees of truth. For instance, given a expression of the form $\rho(C \sqcap D, \geqslant\alpha)$ we can infer both $\rho(C, \geqslant\alpha)$ and $\rho(D, \geqslant\alpha)$, since the semantics of the conjunction is given by the minimum and $\min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \geqslant \alpha$ clearly implies that $C^{\mathcal{I}}(x) \geqslant \alpha$ and $D^{\mathcal{I}}(x) \geqslant \alpha$.

In Łukasiewicz fuzzy logic, the situation is more complicated, since it is not possible to infer the exact degree of truth of the elements that compose a complex concept. However, thanks to Proposition 4.1, we know that they belong to $\mathcal{N}$. Now, the basic idea of the reduction of fuzzy concepts is to build a disjunction over the different possible degrees of truth.

Before proceeding formally, we will illustrate this idea with an example.

Example 4.2. Consider a fuzzy assertion $\tau = \langle a{:}A_1 \sqcap A_2 \geqslant 0.5\rangle$ and $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\}$. Every model $\mathcal{I}$ of τ satisfies $\max\{A_1^{\mathcal{I}}(a^{\mathcal{I}}) + A_2^{\mathcal{I}}(a^{\mathcal{I}}) - 1, 0\} \geqslant 0.5$. Hence, it follows that $A_1^{\mathcal{I}}(a^{\mathcal{I}}) + A_2^{\mathcal{I}}(a^{\mathcal{I}}) - 1 \geqslant 0.5 \iff A_1^{\mathcal{I}}(a^{\mathcal{I}}) + A_2^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant 1.5$. Now, we do not know exactly the degrees of truth of $A_1^{\mathcal{I}}(a^{\mathcal{I}})$ and $A_2^{\mathcal{I}}(a^{\mathcal{I}})$, but they belong to $\mathcal{N}$, so there are six possibilities:

$A_1^{\mathcal{I}}(a^{\mathcal{I}})$	$A_2^{\mathcal{I}}(a^{\mathcal{I}})$	$(A_1 \sqcap A_2)^{\mathcal{I}}(a^{\mathcal{I}})$
0.5	1	0.5
0.75	0.75	0.5
0.75	1	0.75
1	0.5	0.5
1	0.75	0.75
1	1	1

Hence, we can think of a model satisfying a: $(A_{1\geqslant 0.5} \sqcap A_{2\geqslant 1}) \sqcup (A_{1\geqslant 0.75} \sqcap A_{2\geqslant 0.75}) \sqcup (A_{1\geqslant 0.75} \sqcap A_{2\geqslant 1}) \sqcup (A_{1\geqslant 1} \sqcap A_{2\geqslant 0.5}) \sqcup (A_{1\geqslant 1} \sqcap A_{2\geqslant 0.75}) \sqcup (A_{1\geqslant 1} \sqcap A_{2\geqslant 1})$.

The previous example shows that the disjunctions and conjunctions that are introduced in the reduction can be optimized by taking into account the following observations:

Proposition 4.3. *Let B_1, B_2 be two non-fuzzy concepts such that $B_1 \sqsubseteq B_2$. The following hold:*

1. $B_1 \sqcap B_1 \equiv B_1 \sqcup B_1 \equiv B_1$.
2. $B_1 \sqcap B_3 \sqsubseteq B_2 \sqcap B_3$.
3. $B_1 \sqcup B_3 \sqsubseteq B_2 \sqcup B_3$.
4. $B_1 \sqcup B_2 \sqcup B_3 \sqcup \cdots B_m \equiv B_2 \sqcup B_3 \sqcup \cdots B_m$.
5. $B_1 \sqcap B_2 \sqcap B_3 \sqcap \cdots B_m \equiv B_1 \sqcap B_3 \sqcap \cdots aB_m$.

Proof. Trivial. □

Example 4.4. Consider the (non-fuzzy) assertion obtained in Example 4.2 as a result of the reduction. It can be seen that $(A_{1\geqslant 0.75} \sqcap A_{2\geqslant 0.75}) \sqsupseteq (A_{1\geqslant 0.75} \sqcap A_{2\geqslant 1})$ and that $(A_{1\geqslant 1} \sqcap A_{2\geqslant 0.5}) \sqsupseteq (A_{1\geqslant 1} \sqcap A_{2\geqslant 0.75}) \sqsupseteq (A_{1\geqslant 1} \sqcap A_{2\geqslant 1})$. Consequently, the reduction of the axiom is satisfiable iff the following assertion is: a: $(A_{1\geqslant 0.5} \sqcap A_{2\geqslant 1}) \sqcup (A_{1\geqslant 0.75} \sqcap A_{2\geqslant 0.75}) \sqcup (A_{1\geqslant 1} \sqcap A_{2\geqslant 0.5})$.

Concept and role expressions are reduced using mapping ρ, as shown in Table 6.

Mapping ρ deserves some comments. Firstly, it is interesting to remark that $\rho(A, \leqslant \beta) = \neg A_{>\beta}$ is different to $\rho(\neg A, \geqslant \alpha) = \rho(A, \leqslant 1-\alpha) = \neg A_{>1-\alpha}$. Then, due to the restrictions in the definition of the fuzzy KB, some expressions cannot appear during the process:

- Expressions of the form $\rho(A, \geqslant 0)$ and $\rho(A, \leqslant 1)$ cannot appear, because there exist some restrictions on the degree of the axioms in the fuzzy KB. The same also holds for $\top$, $\bot$ and R_A.
- Expressions of the form $\rho(R, \leqslant \beta)$ can only appear in a negated role assertion.
- Expressions of the form $\rho(U, \leqslant \beta)$ cannot appear either.

The case of qualified cardinality restrictions is more involved. We will use a partition is used to simulate the existence of m different individuals (the fillers of the cardinality restriction).

Let $B_1, \ldots, B_m$ be atomic concepts. $B_1, \ldots, B_m$ form a *partition* w.r.t. a fuzzy interpretation $\mathcal{I}$ iff the following conditions hold:

- $\bigcup_{i=1,\ldots,m} \{B_i^{\mathcal{I}}\} = \Delta^{\mathcal{I}}$.
- $B_i^{\mathcal{I}} \cap B_j^{\mathcal{I}} = \emptyset$, for $1 \leqslant i < j \leqslant m$.

For every expression of the form $\rho(\geqslant m\ S.C, \triangleright\gamma)$ that appear in the reduction process, we create m new atomic concepts $B_1, \ldots, B_m$ such they form a partition w.r.t. $\mathcal{I}$. This is achieved by adding the following axioms:

- $\top \sqsubseteq B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m$.
- $B_i \sqcap B_j \sqsubseteq \bot$, for $i < j$, for $1 \leqslant i < j \leqslant m$.

Proposition 4.5. *Let $B_1, \ldots, B_m$ be non empty atomic concepts forming a partition w.r.t. a fuzzy interpretation $\mathcal{I}$, and let b_i denote an individual such that $b_i \in B_i^{\mathcal{I}}$, for all i = 1, …, m. Then, $b_1, \ldots, b_m \in \Delta^{\mathcal{I}}$ are pairwise different individuals.*

Proof. By reduction to absurd. Assume on the contrary that there are two individuals b_i and b_j such that $b_i = b_j$. By assumption, we have that $b_i \in B_i^{\mathcal{I}}$ and $b_j \in B_j^{\mathcal{I}}$. Using that $b_i = b_j$, it follows that $b_i \in B_j^{\mathcal{I}}$. Since $b_i \in B_i^{\mathcal{I}}$ and $b_i \in B_j^{\mathcal{I}}$, $b_i \in (B_i \sqcap B_j)^{\mathcal{I}}$. Finally, using the condition of the partition $B_i \sqcap B_j \sqsubseteq \bot$, it follows that $b_i^{\mathcal{I}} \in \bot^{\mathcal{I}}$, which is absurd. □

Proposition 4.6. *Let $b_1, \ldots, b_m \in \Delta^{\mathcal{I}}$ be pairwise different individuals. Then, there exist atomic concepts $B_1, \ldots, B_m$ forming a partition w.r.t. a fuzzy interpretation $\mathcal{I}$ such that $b_i \in B_i^{\mathcal{I}}$, for all i = 1, …, m.*

Table 6
Mapping of concept and role expressions in fuzzy $\mathcal{SROIQ}$.

x	y	$\rho(x,y)$
$\top$	$\geqslant\alpha$	$\top$
$\top$	$\leqslant\beta$	$\bot$
$\bot$	$\geqslant\alpha$	$\bot$
$\bot$	$\leqslant\beta$	$\top$
A	$\geqslant\alpha$	$A_{\geqslant\alpha}$
A	$\leqslant\beta$	$\neg A_{>\beta}$
$\neg C$	$\bowtie\gamma$	$\rho(C, \bowtie^{-} 1-\gamma)$
$C \sqcap D$	$\geqslant\alpha$	$\bigsqcup_{\gamma_1,\gamma_2}\{\rho(C,\geqslant\gamma_1)\sqcap\rho(D,\geqslant\gamma_2)\}$ for every pair $\gamma_1,\gamma_2\in\mathcal{N}^+$ such that $\gamma_1+\gamma_2=1+\alpha$
$C \sqcap D$	$\leqslant\beta$	$\rho(\neg C\sqcup\neg D,\geqslant 1-\beta)$
$C \sqcup D$	$\geqslant\alpha$	$\rho(C,\geqslant\alpha)\sqcup\rho(D,\ \geqslant\alpha)\bigsqcup\{\gamma_1,\gamma_2\}\{\rho(C,\geqslant\gamma_1)\sqcap\rho(D,\geqslant\gamma_2)\}$ for every pair $\gamma_1,\gamma_2\in\mathcal{N}^+$ such that $\gamma_1+\gamma_2=\alpha$
$C \sqcup D$	$\leqslant\beta$	$\rho(\neg C\sqcap\neg D,\geqslant 1-\beta)$
$\exists R.C$	$\geqslant\alpha$	$\bigsqcup_{\gamma_1,\gamma_2}\{\exists\rho(R,\geqslant\gamma_1).\rho(C,\geqslant\gamma_2)\}$ for every pair $\gamma_1,\gamma_2\in\mathcal{N}^+$ such that $\gamma_1+\gamma_2=1+\alpha$
$\exists R.C$	$\leqslant\beta$	$\rho(\forall R.\neg C,\geqslant 1-\beta)$
$\forall R.C$	$\geqslant\alpha$	$\sqcap_{\gamma_1,\gamma_2}\{\forall\rho(R,\geqslant\gamma_1).\rho(C,\geqslant\gamma_2)\}$ for every pair $\gamma_1,\gamma_2\in\mathcal{N}^+$ such that $\gamma_1=\gamma_2+1-\alpha$
$\forall R.C$	$\leqslant\beta$	$\rho(\exists R.\neg C,\geqslant 1-\beta)$
$\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$	$\bowtie\gamma$	$\{o_i \mid \alpha_i\bowtie\gamma, 1\leqslant i\leqslant n\}$
$\geqslant m\ S.C$	$\geqslant\alpha$	$\bigsqcup_{\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2}\{\exists\rho(S,\geqslant\gamma_1^1).(B_1\sqcap\rho(C,\geqslant\gamma_1^2))\sqcap \ldots\sqcap\exists\rho(S,\geqslant\gamma_m^1).(B_m\sqcap\rho(C,\geqslant\gamma_m^2))\}$ for every combination of $\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\ \gamma_m^2\in\mathcal{N}^+$ such that $\gamma_i^1+\gamma_i^2=1+\alpha$, for $i=\{1,\ldots,m\}$
$\geqslant m\ S.C$	$\leqslant\beta$	$\neg\Big[\bigsqcup_{\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2}\{\exists\rho(S,\geqslant\gamma_1^1).(B_1\sqcap\rho(C,\geqslant\gamma_1^2))\sqcap \ldots\sqcap\exists\rho(S,\geqslant\gamma_m^1).(B_m\sqcap\rho(C,\geqslant\gamma_m^2))\}\Big]$ for every combination of $\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2\in\mathcal{N}^+$ such that (i) $\gamma_i^1+\gamma_i^2>1+\beta$, for $i=\{1,\ldots,m\}$, and (ii) $\nexists\gamma\in\mathcal{N}^+$ such that $\gamma<\gamma_i^1$ and $\gamma+\gamma_i^2>1+\beta$, and (iii) $\nexists\gamma\in\mathcal{N}^+$ such that $\gamma<\gamma_i^2$ and $\gamma_i^1+\gamma>1+\beta$
$\leqslant n\ S.C$	$\geqslant\alpha$	$\rho(\neg(\geqslant n+1S.C),\geqslant\alpha)$
$\leqslant n\ S.C$	$\leqslant\beta$	$\rho(\neg(\geqslant n+1S.C),\leqslant\beta)$
$\exists S.\mathtt{Self}$	$\geqslant\alpha$	$\exists\rho(S,\geqslant\ \alpha).Self$
$\exists S.\mathtt{Self}$	$\leqslant\beta$	$\neg\exists\rho(S,>\beta).Self$
R_A	$\geqslant\alpha$	$R_{A\geqslant\alpha}$
R_A	$\leqslant\beta$	$\neg R_{A>\beta}$
R^-	$\geqslant\alpha$	$\rho(R,\geqslant\alpha)^-$
R^-	$\leqslant\beta$	$\rho(R,\leqslant\beta)^-$
U	$\geqslant\alpha$	U

Proof. It is trivial to assume the existence of m atomic concepts verifying $\bigcup_{i=1,\ldots,m}\{B_i^{\mathcal{I}}\}=\Delta^{\mathcal{I}}$ and $b_i\in B_i^{\mathcal{I}}$. Since $b_i\neq b_j$ for $1\leqslant i<j\leqslant m$, the condition $B_i^{\mathcal{I}}\cap B_j^{\mathcal{I}}=\emptyset$ can also be satisfied. □

Then, $\rho(\geqslant m\ S.C,\triangleright\gamma)$ is transformed into a conjunction of m expressions of the form $(\geqslant 1\rho(S\geqslant\gamma_i^1).(\rho(C\geqslant\gamma_i^2)\sqcap B_i))$, each of them simulating one of the mutually different m fillers of the cardinality restriction, and in such a way that the degrees γ_i^1,γ_i^2 satisfy the semantics of the constructor.

The reduction $\rho(\geqslant m\ S.C,\triangleleft\gamma)$ is based on the equivalence:

$$\rho(\geqslant mS.C,\triangleleft\gamma)\equiv\neg\rho(\geqslant mS.C,\neg\triangleleft\gamma)$$

Finally, the reduction of $\rho(\leqslant m\ S.C,\bowtie\gamma)$ is based on the equivalence:

$$(\leqslant nS.C)\equiv\neg(\geqslant n+1S.C)$$

Axioms are reduced as in Table 7, where κ maps fuzzy ABox, TBox, and RBox axiom into non-fuzzy ABox, TBox, and RBox axioms, respectively.

The reader may be tempted to think that the reduction of a fuzzy GCI should take into account every pair $\gamma_1,\gamma_2\in\mathcal{N}^+$ such that $\gamma_1\leqslant\gamma_2+1-\alpha$. However, the additional axioms are superfluous as the following example illustrates. The case of fuzzy RIAs is similar.

Example 4.7. Consider a fuzzy GCI $\langle A_1\sqsubseteq A_2\geqslant 0.5\rangle$ and $\mathcal{N}=\{0,0.25,0.5,0.75,1\}$. For every individual x of the interpretation domain, it follows that $A_1^{\mathcal{I}}(x)\Rightarrow A_2^{\mathcal{I}}(x)\geqslant 0.5$ and thus $1-A_1^{\mathcal{I}}(x)+A_2^{\mathcal{I}}(x)\geqslant 0.5$. This introduces several possibilities:

$A_1^{\mathcal{I}}(x) \setminus A_2^{\mathcal{I}}(x)$	0	0.25	0.5	0.75	1
0	✓	✓	✓	✓	✓
0.25	✓	✓	✓	✓	✓
0.5	✓	✓	✓	✓	✓
0.75		✓	✓	✓	✓
1			✓	✓	✓

It is easy to see that $A_1^{\mathcal{I}}(x) \geqslant 0.75$ implies that $A_2^{\mathcal{I}}(x) \geqslant 0.25$, and that $A_1^{\mathcal{I}}(x) \geqslant 1$ implies that $A_2^{\mathcal{I}}(x) \geqslant 0.5$. This restriction is hence equivalent to this couple of GCIs: $\rho(A_1, \geqslant 0.75) \sqsubseteq \rho(A_2, \geqslant 0.25)$, and $\rho(A_1, \geqslant 1) \sqsubseteq \rho(A_2, \geqslant 0.5)$.

As we see, we take $\gamma_1, \gamma_2 \in \mathcal{N}^+$ verifying $\gamma_1 = \gamma_2 + 1 - \alpha = \gamma_2 + 0.5$.

Without loss of generality (see Proposition 3.4), we assume that negated role assertions, transitive and symmetric role axioms do not appear in the fuzzy KB. However, we do include the reduction of irreflexive role axioms because it is more efficient than using the equivalence in Proposition 3.4.

We note $\kappa(\mathcal{A})$ (resp. $\kappa(\mathcal{T}), \kappa(\mathcal{R})$) the union of the reductions of every axiom in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$). To be precise, the reduction of fuzzy GCIs and RIAs should be noted as $\kappa(\tau, \mathcal{N})$, and the reduction of the fuzzy TBox and RBox as $\kappa(\mathcal{T}, \mathcal{N})$ and $\kappa(\mathcal{R}, \mathcal{N})$, respectively. For the sake of simplicity we omit $\mathcal{N}$ since it is clear from the context.

Let $\mathsf{crisp}(\mathcal{K})$ denote the reduction of a fuzzy ontology $\mathcal{K}$. A fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced into a KB $\mathsf{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{N}) \cup \kappa(\mathcal{T}), R(\mathcal{N}) \cup \kappa(\mathcal{R}) \rangle$.

4.3. Examples

Now we will illustrate how the reduction works using two examples.

Example 4.8. Let us a consider a fuzzy KB $\mathcal{K} = \{\langle a : \forall R.(C \sqcap D) \geqslant 0.75\rangle, \langle (a,b) : R \geqslant 0.75\rangle, \langle b : \neg C \geqslant 0.75\rangle\}$ and assume a set of degrees of truth $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\} (n = 4)$. Note that the TBox and the RBox are empty.

This fuzzy KB is clearly unsatisfiable. From the third assertion it follows that $C^{\mathcal{I}}(b^{\mathcal{I}}) \leqslant 0.25$, and it can be seen that this implies that $(C \sqcap D)^{\mathcal{I}}(b^{\mathcal{I}}) = \max\{C^{\mathcal{I}} + D^{\mathcal{I}} - 1, 0\} \leqslant 0.25$. But from the two former assertions it follows that every fuzzy interpretation $\mathcal{I}$ has to satisfy $(C \sqcap D)^{\mathcal{I}}(b^{\mathcal{I}}) \geqslant 0.5$, which is a contradiction.

Now, let us compute the non-fuzzy representation of $\mathcal{K}$. Firstly, we create some new non-fuzzy atomic concepts associated to the set of atomic fuzzy concepts and some new non-fuzzy atomic roles associated to the set of atomic fuzzy roles:

- New concepts: $C_{>0}$, $C_{\geqslant 0.25}$, $C_{>0.25}$, $C_{\geqslant 0.5}$, $C_{>0.5}$, $C_{\geqslant 0.75}$, $C_{>0.75}$, $C_{\geqslant 1}$, $D_{>0}$, $D_{\geqslant 0.25}$, $D_{>0.25}$, $D_{\geqslant 0.5}$, $D_{>0.5}$, $D_{\geqslant 0.75}$, $D_{>0.75}$, $D_{\geqslant 1}$.
- New roles: $R_{\geqslant 0.25}$, $R_{\geqslant 0.5}$, $R_{\geqslant 0.75}$, $R_{\geqslant 1}$.

Now we create some new axioms to preserve the semantics of these elements:

- $T(\mathcal{N}) = \{C_{\geqslant 1} \sqsubseteq C_{>0.75}, C_{>0.75} \sqsubseteq C_{\geqslant 0.75}, C_{\geqslant 0.75} \sqsubseteq C_{>0.5}, C_{>0.5} \sqsubseteq C_{\geqslant 0.5}, C_{\geqslant 0.5} \sqsubseteq C_{>0.25}, C_{>0.25} \sqsubseteq C_{\geqslant 0.25}, C_{\geqslant 0.25} \sqsubseteq C_{>0}, D_{\geqslant 1} \sqsubseteq D_{>0.75}, D_{>0.75} \sqsubseteq D_{\geqslant 0.75}, D_{\geqslant 0.75} \sqsubseteq D_{>0.5}, D_{>0.5} \sqsubseteq D_{\geqslant 0.5}, D_{\geqslant 0.5} \sqsubseteq D_{>0.25}, D_{>0.25} \sqsubseteq D_{\geqslant 0.25}, D_{\geqslant 0.25} \sqsubseteq D_{>0}\}$.
- $R(\mathcal{N}) = \{R_{\geqslant 1} \sqsubseteq R_{\geqslant 0.75}, R_{\geqslant 0.75} \sqsubseteq R_{\geqslant 0.5}, R_{\geqslant 0.5} \sqsubseteq R_{\geqslant 0.25}\}$.

Table 7
Reduction of the axioms.

$\kappa(\langle a{:}C \geqslant \alpha\rangle)$	$a{:}\rho(C, \geqslant \alpha)$
$\kappa(\langle a{:}C \leqslant \beta\rangle)$	$a{:}\rho(C, \leqslant \beta)$
$\kappa(\langle (a,b){:}R \geqslant \alpha\rangle)$	$(a,b){:}\rho(R, \geqslant \alpha)$
$\kappa(\langle (a,b){:}R \leqslant \beta\rangle)$	$(a,b){:}\rho(R, \leqslant \alpha)$
$\kappa(\langle a \neq b\rangle)$	$a \neq b$
$\kappa(\langle a = b\rangle)$	$a = b$
$\kappa(\langle C \sqsubseteq D \geqslant \alpha\rangle)$	$\bigcup_{\gamma_1,\gamma_2}\{\rho(C, \geqslant \gamma_1) \sqsubseteq \rho(D, \geqslant \gamma_2)\}$
	for every pair $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 = \gamma_2 + 1 - \alpha$
$\kappa(\langle R_1 \ldots R_n \sqsubseteq R \geqslant \alpha\rangle)$	$\bigcup_{\gamma_1,\ldots,\gamma_n}\{\rho(R_1, \geqslant \gamma_1) \ldots \rho(R_n, \geqslant \gamma_n) \sqsubseteq \rho(R, \geqslant \gamma_{n+1})\}$
	for every combination $\gamma_1, \ldots, \gamma_{n+1} \in \mathcal{N}^+$
	such that $\gamma_1 + \ldots + \gamma_n = \gamma_{n+1} + n - \alpha$
$\kappa(\mathtt{dis}(S_1, S_2))$	$\mathtt{dis}(\rho(S_1, > 0), \rho(S_2, > 0))$
$\kappa(\mathtt{ref}(R))$	$\mathtt{ref}(\rho(R, \geqslant 1))$
$\kappa(\mathtt{irr}(S))$	$\mathtt{irr}(\rho(S, > 0))$
$\kappa(\mathtt{asy}(S))$	$\mathtt{asy}(\rho(S, > 0)$

Now we are ready to compute $\kappa(\mathcal{A})$, including the reduction of the three fuzzy assertions in the fuzzy KB, that is:

- $\kappa(\langle(a, b){:}R \geqslant 0.75\rangle) = (a, b){:}\rho(R, \geqslant 0.75) = (a, b){:}R_{\geqslant 0.75}$.
- $\kappa(\langle b : \neg C \geqslant 0.75\rangle) = b : \rho(\neg C, \geqslant 0.75) = b : \neg C_{>0.25}$.
- $\kappa(\langle a{:}\forall R. (C \sqcap D) \geqslant 0.75\rangle) = a{:}\rho(\forall R. (C \sqcap D), \geqslant 0.75) = a{:}\{\forall\rho(R, \geqslant 0.5).\rho(C \sqcap D, \geqslant 0.25) \sqcap \forall\rho(R, \geqslant 0.75).\rho(C \sqcap D, \geqslant 0.5) \sqcap \forall\rho(R, \geqslant 1).\rho(C \sqcap D, \geqslant 0.75)\}$, where:
  - $\rho(R, \geqslant 0.5) = R_{\geqslant 0.5}$.
  - $\rho(C \sqcap D, \geqslant 0.25) = (C_{\geqslant 0.25} \sqcap D_{\geqslant 1}) \sqcup (C_{\geqslant 0.5} \sqcap D_{\geqslant 0.75}) \sqcup (C_{\geqslant 0.75} \sqcap D_{\geqslant 0.5}) \sqcup (C_{\geqslant 1} \sqcap D_{\geqslant 0.25})$.
  - $\rho(R, \geqslant 0.75) = R_{\geqslant 0.75}$.
  - $\rho(C \sqcap D, \geqslant 0.5) = (C_{\geqslant 0.5} \sqcap D_{\geqslant 1}) \sqcup (C_{\geqslant 0.75} \sqcap D_{\geqslant 0.75}) \sqcup (C_{\geqslant 1} \sqcap D_{\geqslant 0.5})$.
  - $\rho(R, \geqslant 1) = R_{\geqslant 1}$.
  - $\rho(C \sqcap D, \geqslant 0.75) = (C_{\geqslant 0.75} \sqcap D_{\geqslant 1}) \sqcup (C_{\geqslant 1} \sqcap D_{\geqslant 0.75})$.

It can be seen that the (non-fuzzy) KB $\mathsf{crisp}(\mathcal{K}) = \langle\kappa(\mathcal{A}), T(\mathcal{N}), R(\mathcal{N})\rangle$ is unsatisfiable.

Next we illustrate how to reduce cardinality restrictions, which are more involved.

Example 4.9. Let us a consider a fuzzy KB $\mathcal{K} = \{\langle a : \leqslant 1S.C \geqslant 0.75\rangle, \langle(a,b) : S \geqslant 0.75\rangle, \langle(a,c) : S \geqslant 0.75\rangle, \langle b : C \geqslant 0.75\rangle, \langle c : C \geqslant 0.75\rangle, \langle b \neq c\rangle\}$ and assume a set of degrees of truth $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\}$ ($n = 4$). Note that the TBox and the RBox are empty.

This fuzzy KB is clearly unsatisfiable. From the first assertion it follows that $\min\{S^{\mathcal{I}}(a,b) \otimes C^{\mathcal{I}}(b), S^{\mathcal{I}}(a,c) \otimes C^{\mathcal{I}}(c)\} \Rightarrow (b = c) \geqslant 0.75$. This is true in two cases:

- $b = c$, or
- $\min\{S^{\mathcal{I}}(a,b) \otimes C^{\mathcal{I}}(b), S^{\mathcal{I}}(a,c) \otimes C^{\mathcal{I}}(c)\} \leqslant 0.25$.

The first possibility is clearly in contradiction with the last assertion of the fuzzy KB. Moreover, assertions 2, 3, 4 and 5 imply that $\min\{S^{\mathcal{I}}(a,b) \otimes C^{\mathcal{I}}(b), S^{\mathcal{I}}(a,c) \otimes C^{\mathcal{I}}(c)\} = \min\{0.75 \otimes 0.75, 0.75 \otimes 0.75\} = 0.5$, which is in contradiction with the second possibility. Hence, the fuzzy KB is unsatisfiable.

Now, let us compute the non-fuzzy representation of $\mathcal{K}$. Firstly, we create some new non-fuzzy atomic concepts associated to the set of atomic fuzzy concepts and some new non-fuzzy atomic roles associated to the set of atomic fuzzy roles:

- *New concepts*: $C_{>0}$, $C_{\geqslant 0.25}$, $C_{>0.25}$, $C_{\geqslant 0.5}$, $C_{>0.5}$, $C_{\geqslant 0.75}$, $C_{>0.75}$, $C_{\geqslant 1}$.
- *New roles*: $S_{\geqslant 0.25}$, $S_{\geqslant 0.5}$, $S_{\geqslant 0.75}$, $S_{\geqslant 1}$.

Now we create some new axioms to preserve the semantics of these elements:

- $T(\mathcal{N}) = \{C_{\geqslant 1} \sqsubseteq C_{>0.75}, C_{>0.75} \sqsubseteq C_{\geqslant 0.75}, C_{\geqslant 0.75} \sqsubseteq C_{>0.5}, C_{>0.5} \sqsubseteq C_{\geqslant 0.5}, C_{\geqslant 0.5} \sqsubseteq C_{>0.25}, C_{>0.25} \sqsubseteq C_{\geqslant 0.25}, C_{\geqslant 0.25} \sqsubseteq C_{>0}\}$.
- $R(\mathcal{N}) = \{S_{\geqslant 1} \sqsubseteq S_{\geqslant 0.75}, S_{\geqslant 0.75} \sqsubseteq S_{\geqslant 0.5}, S_{\geqslant 0.5} \sqsubseteq S_{\geqslant 0.25}\}$.

Now we are ready to compute $\kappa(\mathcal{A})$, including the reduction of the six fuzzy assertions in the fuzzy KB, that is:

- $\kappa(\langle a : (\leqslant 1S.C) \geqslant 0.75\rangle) = a : \rho(\neg(\geqslant 2S.C), \geqslant 0.75) = a : \rho(\geqslant 2S.C, \leqslant 0.25) =$

$$a : \neg\begin{pmatrix} [\exists\rho(S, \geqslant 0.5).(B_1 \sqcap \rho(C, \geqslant 1)) \sqcap \exists\rho(S, \geqslant 0.5).(B_2 \sqcap \rho(C, \geqslant 1))] \sqcup \\ [\exists\rho(S, \geqslant 0.5).(B_1 \sqcap \rho(C, \geqslant 1)) \sqcap \exists\rho(S, \geqslant 0.75).(B_2 \sqcap \rho(C, \geqslant 0.75))] \sqcup \\ [\exists\rho(S, \geqslant 0.5).(B_1 \sqcap \rho(C, \geqslant 1)) \sqcap \exists\rho(S, \geqslant 1).(B_2 \sqcap \rho(C, \geqslant 0.5))] \sqcup \\ [\exists\rho(S, \geqslant 0.75).(B_1 \sqcap \rho(C, \geqslant 0.75)) \sqcap \exists\rho(S, \geqslant 0.5).(B_2 \sqcap \rho(C, \geqslant 1))] \sqcup \\ [\exists\rho(S, \geqslant 0.75).(B_1 \sqcap \rho(C, \geqslant 0.75)) \sqcap \exists\rho(S, \geqslant 0.75).(B_2 \sqcap \rho(C, \geqslant 0.75))] \sqcup \\ [\exists\rho(S, \geqslant 0.75).(B_1 \sqcap \rho(C, \geqslant 0.75)) \sqcap \exists\rho(S, \geqslant 1).(B_2 \sqcap \rho(C, \geqslant 0.5))] \sqcup \\ [\exists\rho(S, \geqslant 1).(B_1 \sqcap \rho(C, \geqslant 0.5)) \sqcap \exists\rho(S, \geqslant 0.5).(B_2 \sqcap \rho(C, \geqslant 1))] \sqcup \\ [\exists\rho(S, \geqslant 1).(B_1 \sqcap \rho(C, \geqslant 0.5)) \sqcap \exists\rho(S, \geqslant 0.75).(B_2 \sqcap \rho(C, \geqslant 0.75))] \sqcup \\ [\exists\rho(S, \geqslant 1).(B_1 \sqcap \rho(C, \geqslant 0.5)) \sqcap \exists\rho(S, \geqslant 1).(B_2 \sqcap \rho(C, \geqslant 0.5))] \end{pmatrix}.$$

The reduction of this concept expression also introduces two new atomic concepts B_1, B_2 together with the axioms: $B_1 \sqcap B_2 \sqsubseteq \bot$, $\top \sqsubseteq B_1 \sqcup B_2$.

- $\kappa(\langle(a, b){:}S \geqslant 0.75\rangle) = (a, b){:}\rho(S, \geqslant 0.75) = (a, b){:}S_{\geqslant 0.75}$.
- $\kappa(\langle b{:}C \geqslant 0.75\rangle) = b{:}\rho(C, \geqslant 0.75) = b{:}C_{\geqslant 0.75}$.
- $\kappa(\langle(a, c){:}S \geqslant 0.75\rangle) = (a, b){:}\rho(S, \geqslant 0.75) = (a, c){:}S_{\geqslant 0.75}$.
- $\kappa(\langle c{:}C \geqslant 0.75\rangle) = b{:}\rho(C, \geqslant 0.75) = c{:}C_{\geqslant 0.75}$.
- $\kappa(\langle b \neq c\rangle) = b \neq c$.

It can be seen that the (non-fuzzy) KB $\mathsf{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{N}), R(\mathcal{N}) \rangle$ is unsatisfiable.

4.4. Properties of the reduction

Firstly, we highlight that the reduction preserves simplicity of the roles and regularity of the RIAs.

Correctness. The reduction is reasoning preserving and, since satisfiability testing in classical $\mathcal{SROIQ}$ is decidable [30] and the mapping is finite it follows that:

Theorem 4.10. *The satisfiability problem in* $Ł_n\mathcal{SROIQ}$ *with truth space* $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\right\} (n \in \mathbb{N})$ *is decidable. Furthermore, a* $Ł_n\mathcal{SROIQ}$ *fuzzy KB* $\mathcal{K}$ *is satisfiable iff* $\mathsf{crisp}(\mathcal{K})$ *is satisfiable.*

Proof. See Appendix. □

Complexity. The *depth of a fuzzy concept* is inductively defined as follows:

- $\mathsf{depth}(A) = \mathsf{depth}(\top) = \mathsf{depth}(\bot) = \mathsf{depth}(\exists S.Self) = \mathsf{depth}(\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}) = 1$.
- $\mathsf{depth}(\neg C) = \mathsf{depth}(\forall R.C) = \mathsf{depth}(\exists R.C) = \mathsf{depth}(\geqslant mS.C) = \mathsf{depth}(\leqslant nS.C) = 1 + \mathsf{depth}(C)$.
- $\mathsf{depth}(C \sqcap D) = \mathsf{depth}(C \sqcup D) = 1 + max\ \{\mathsf{depth}(C), \mathsf{depth}(D)\}$.

The depth of a non-fuzzy concept is defined analogously. It is easy to see that:

- A concept C *without* number restrictions of depth k transforms in the worst case into an expression of size $\mathcal{O}(|C||\mathcal{N}|^k)$. For instance, for C being $\forall R.(\forall P.(\forall Q.A))$, we get an expression of size $\mathcal{O}(|C||\mathcal{N}|^3)$ ($k = 3$).
- A concept C *with* number restrictions of depth k transforms in the worst case into an expression of size $\mathcal{O}(m^{k-1}|C||\mathcal{N}|^{(k-1)m})$, where m is the maximal number restriction occurring in C. For instance, for C being $(\geqslant m_1 R.(\geqslant m_2 P.A))$, we get an expression of size $\mathcal{O}(m^2|C||\mathcal{N}|^{2m})$ ($k = 3$, $m = \max(m_1, m_2)$).
- In order to preserve the semantics of the new atomic concept and roles, we are also introducing some new non-fuzzy axioms:

$$|T(\mathcal{N})| = (2 \cdot (|\mathcal{N}| - 1) - 1) \cdot |\mathsf{A}|$$

$$|R(\mathcal{N})| = (2 \cdot (|\mathcal{N}| - 1) - 1) \cdot |\mathsf{R}|$$

- The reduction of qualified cardinality restrictions $\rho(\geqslant mS.C, \triangleright\gamma)$ also introduces $\binom{m}{2} + 1$ GCIs.
- Most of the axioms of the fuzzy KB generate one non-fuzzy axiom, but some of them (fuzzy GCIs and fuzzy RIAs) generate several non-fuzzy axioms:

$$|\kappa(\mathcal{T})| \leqslant 2 \cdot (|\mathcal{N}| - 1) \cdot |\mathcal{T}|$$

$$|\kappa(\mathcal{R})| \leqslant 2 \cdot (|\mathcal{N}| - 1) \cdot |\mathcal{R}|$$

All in all, the size of the resulting KB is $\mathcal{O}(|\mathcal{K}||\mathcal{N}|^k)$ in case *no* number restriction occurs in $\mathcal{K}$, where k is the maximal depth of the concepts appearing in the fuzzy KB, while otherwise is $\mathcal{O}(m^{k-1}|\mathcal{K}||\mathcal{N}|^{(k-1)m})$, where m is the maximal number restriction occurring in $\mathcal{K}$.

We recall that under Zadeh fuzzy logic, the size of the resulting KB is $\mathcal{O}(|\mathcal{K}|^2)$ [61,7,5]. In our case we need to generate more and more complex axioms, because we cannot infer the exact values of the elements which take part of a complex concept, so we need to build disjunctions or conjunctions over all possible degrees of truth.

Modularity. An interesting property of the procedure is that the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included. From an implementation point of view, this property allows to compute the reduction of the ontology off-line and update $\kappa(\mathcal{K})$ incrementally.

Theorem 4.11. *Let* $\mathcal{K}$ *be a* $Ł_n\mathcal{SROIQ}$ *fuzzy knowledge base involving a set of fuzzy atomic concepts* **A** *and a set of atomic roles* **R**; *let* $\mathcal{N} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\} (n \in \mathbb{N})$; *and let* τ *be a* $Ł_n\mathcal{SROIQ}$ *axiom such that:*

1. *for every atomic concept* A *which appears in* τ, $A \in \mathbf{A}$.
2. *for every atomic role* R_A *which appears in* τ, $R_A \in \mathbf{R}$.
3. *if* γ *appears in* τ, *then* $\gamma \in \mathcal{N}$.

Then, $\mathsf{crisp}(\mathcal{K} \cup \tau) = \mathsf{crisp}(\mathcal{K}) \cup \kappa(\tau)$.

Proof. The proof is trivial from the following observations:

- Every axiom is reduced to a combination of new non-fuzzy elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- τ does not introduce atomic concepts, atomic roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others. □

The theorem assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often. These are reasonable assumptions because ontologies do not usually change once that their development has finished. Moreover, we have assumed a fixed set of the degrees of truth $\mathcal{N}$.

5. Related work

Since the first work of J. Yen in 1991 [65], an important number of fuzzy extensions to DLs can be found in the literature [41]. We would also like to stress the existence of fuzzy rough DLs [13,33,34] (which extend rough DLs [40,19,51,21,35]) and fuzzy possibilistic DLs [6].

In this section we will concentrate on the state of the art on the fuzzy logics considered in the framework of fuzzy DLs which are different from Zadeh fuzzy logic, and in the representation of fuzzy DLs using non-fuzzy DLs.

5.1. Fuzzy logics in fuzzy DLs

While most of the works restrict themselves to Zadeh fuzzy logic, a few other works consider Łukasiewicz fuzzy logic. Straccia and coworkers [58,62,59,12] propose a reasoning solution, which is based on a mixture of tableau rules and Mixed Integer Linear Programming (MILP) optimization problems. These works are implemented in the FUZZYDL reasoner [11].[4] Habiballa [25] considers a fuzzy extension of $\mathcal{ALC}$ extended with role negation, top role and bottom role, presenting a novel reasoning algorithm based on resolution, as well as an implementation (GERDS). Another implementation based on resolution (YADLR) has been recently presented [38].

A proposal for a product t-norm-based fuzzy DL has also been presented [9], using Product logic but replacing Gödel negation with Łukasiewicz negation.

A Gödel fuzzy DL has been presented in [8]. Previously, [7] considered Gödel implication, but only in the semantics of GCIs and RIAs.

There are also some attempts to reason with arbitrary continuous t-norms. Hájek reported a reasoning algorithm based on a reduction to fuzzy propositional logic [27]. The authors of the present paper have also recently presented a reasoning algorithm for fuzzy DLs under arbitrary continuous t-norms extended with Łukasiewicz negation, based on a combination tableau rules and Mixed Integer Non Linear Programming (MINLP) optimization problems [10]. Both of these works are restricted to the witnessed models of fuzzy $\mathcal{ALC}$ without fuzzy GCIs.

5.2. Non-fuzzy representations for fuzzy DLs

The first effort in this direction is a reasoning preserving procedure for fuzzy $\mathcal{ALCH}$ [61]. A similar work from him considers fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice [57], therefore supporting quantitative reasoning (by using the interval [0,1]) and qualitative reasoning (e.g. by relying on a set {`false`, `likelyfalse`, `unknown`, `likelytrue`, `true`}). Bobillo et al. widened the former work of Straccia to $\mathcal{SHOIN}$ and allowed fuzzy GCIs, but with a semantics given by KD implication [4]. Stoilos et al. extended this work and considered the reduction of an extension of fuzzy $\mathcal{SHOIN}$ with additional role axioms: general RIAs, reflexive, asymmetric and role disjointness axioms [53]. It is not a reduction of fuzzy $\mathcal{SROIQ}$ (not even $\mathcal{SROIN}$) because they do not show how to reduce the universal role, qualified cardinality restrictions, local reflexivity concepts in expressions of the form $\rho(\exists S.\texttt{Self}, \lhd\gamma)$ nor negative role assertions. Moreover, GCIs and RIAs are forced to be either true or false. Bobillo et al. extended this work providing a non-fuzzy representation of full $\mathcal{SROIQ}$ with fuzzy GCIs and RIAs [7].

A different approach considers a family of fuzzy DLs using α-cuts as atomic concept and roles [43]. The approach is slightly different to ours because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce a fuzzy $\mathcal{ALCQ}$ ABox [42] and a fuzzy $\mathcal{ALCH}$ concept [36] to their non-fuzzy versions. Nevertheless, both of these works assume an empty TBox. Finally, [20] combines possibilistic and fuzzy logics in the context of Description Logics (more concretely, in $\mathcal{ALCIN}(\circ)$). Interestingly, they also propose to represent every fuzzy set using two sets (its support and its core) and comment the possibility of using more sets, in order to have a more refined representation. Although for some applications this representation may be enough, there is a loss of information that does not occur in our approach.

[4] <http://www.straccia.info/software/fuzzyDL/fuzzyDL.html>.

All this previous work has been restricted to Zadeh fuzzy logic, with the exception of a non-fuzzy representation for Gödel fuzzy $\mathcal{SROIQ}$ [8], and of a previous version of the current paper, which considers Ł-$\mathcal{ALCHOI}$ [14].

Finally, non-fuzzy representations for two components of fuzzy DLs (which are independent of the particular logic) have been proposed. [5] considers the reduction of fuzzy concrete domains, whereas [8] deals with modified fuzzy concepts and roles.

6. Conclusions and future work

In this paper we have shown the decidability of a fuzzy extension of $\mathcal{SROIQ}$ under the semantics of finitely many-valued Łukasiewicz fuzzy logic, assuming a fixed set of allowed degrees of truth. We have provided a reasoning algorithm based on a reduction to $\mathcal{SROIQ}$. Together with the non-fuzzy representation of fuzzy concrete domains proposed in [5], this means the possibility to reason with Łukasiewicz fuzzy OWL 2.

Providing non-fuzzy representations for fuzzy DLs means an important step towards the possibility of dealing with imprecise and vague knowledge in DLs, since it relies on existing languages and tools. This approach has several advantages:

- We can continue using standard languages with a lot of resources available, avoiding the need (and cost) of adapting them to the new fuzzy language.
- We may continue using existing DL reasoners, which is important because current fuzzy DL reasoners cannot support a fuzzy extension of OWL 2 under Łukasiewicz fuzzy logic (FUZZYDL supports fuzzy OWL-Lite so far).

Our work is more general than previous approaches which provide non-fuzzy representations of fuzzy DLs under Zadeh fuzzy logic. However, from a practical point of view, the size of the resulting KB is much more complex in this case, so the practical feasibility of this approach has to be empirically verified.

In future work we plan to implement the proposed reduction, studying if it can be optimized in some particular common situations.

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Appendix A. Proof of Thereom 4.10

Proof. We will show the proof for the only-if direction. From $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ satisfying every axiom in $\mathcal{K}$. Now, it is possible to build a (non-fuzzy) interpretation $\mathcal{I}_{\mathcal{C}} = \{\Delta^{\mathcal{I}_c}, \cdot^{\mathcal{I}_c}\}$ in the following way:

- $\Delta^{\mathcal{I}_c} = \Delta^{\mathcal{I}}$.
- $a^{\mathcal{I}_c} = a^{\mathcal{I}}$, for all $a \in \Delta^{\mathcal{I}}$.
- $A_{\geqslant\alpha}^{\mathcal{I}_c} = \{x \in \Delta^{\mathcal{I}} | A^{\mathcal{I}}(x) \geqslant \alpha\}$, for each $A \in \mathbf{A}$ and $\alpha \in \mathcal{N} \setminus \{0\}$.
- $A_{>\beta}^{\mathcal{I}_c} = \{x \in \Delta^{\mathcal{I}} | A^{\mathcal{I}}(x) > \beta\}$, for each $A \in \mathsf{A}, \beta \in \mathcal{N} \setminus \{1\}$.
- $R_{A\geqslant\alpha}^{\mathcal{I}_c} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | R_A^{\mathcal{I}}(x, y) \geqslant \alpha\}$, for each $R_A \in \mathsf{R}, \alpha \in \mathcal{N} \setminus \{0\}$.
- $R_{A>\beta}^{\mathcal{I}_c} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | R_A^{\mathcal{I}}(x, y) > \beta\}$, for each $R_A \in \mathbf{R}$, $\beta \in \mathcal{N} \setminus \{1\}$.

Now, it can be shown that $\mathcal{I}_{\mathcal{C}}$ satisfies every axiom in $\mathsf{crisp}(\mathcal{K})$. For every axiom $\tau \in \mathcal{K}$, there are several cases:

1. τ is an *inequality assertion*. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_c} \neq b^{\mathcal{I}_c}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a \neq b \rangle \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a \neq b \rangle)$.
2. τ is an *equality assertion*. Assume that $\mathcal{I} \models \langle a = b \rangle$. Then, $a^{\mathcal{I}} = b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_c} = b^{\mathcal{I}_c}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a = b \rangle \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a = b \rangle)$.
3. τ is a *role assertion*. Assume that $\mathcal{I} \models \langle (a, b) : R \bowtie \gamma \rangle$. We show, by induction on the structure of roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R \bowtie \gamma \rangle)$.
   - *Atomic role*. Assume that $\mathcal{I} \models \langle (a, b) : R_A \geqslant \alpha \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geqslant \alpha$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in R_{A\geqslant\alpha}^{\mathcal{I}_c}$. By definition of $\rho, (a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (\rho(R_A, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(R_A, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R_A \geqslant \alpha \rangle)$.
     Now assume that $\mathcal{I} \models \langle (a, b) : R_A \leqslant \beta \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leqslant \beta$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \notin (R_{A>\beta})^{\mathcal{I}_c}$ and hence $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (\neg R_{A>\beta})^{\mathcal{I}_c}$. By definition of $\rho, (a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (\rho(R_A, \leqslant \beta))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(R_A, \leqslant \beta) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R_A \leqslant \beta \rangle)$.
   - *Inverse role*. Assume that $\mathcal{I} \models \langle (a, b) : R^- \bowtie \gamma \rangle$. Then, $R^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}_c}, a^{\mathcal{I}_c}) \in (\rho(R, \bowtie \gamma))^{\mathcal{I}_c}$. Consequently, $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in ((\rho(R, \bowtie \gamma))^{\mathcal{I}_c})^- \iff \mathcal{I}_{\mathcal{C}} \models (a, b) \in \rho(R, \bowtie \gamma)^- \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R^- \bowtie \gamma \rangle)$.

- *Universal role*. Assume that $\mathcal{I} \models \langle (a,b) : U \geqslant \alpha \rangle$. Then, $U^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1 \geqslant \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in \Delta^{\mathcal{I}_c} \times \Delta^{\mathcal{I}_c}$ and consequently $(a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in U^{\mathcal{I}_c} \iff (a^{\mathcal{I}_c}, b^{\mathcal{I}_c}) \in (\rho(U, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(U, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : U \geqslant \alpha \rangle)$. The case $\mathcal{I} \models \langle (a,b) : U \leqslant \beta \rangle$ is similar.

4. τ is a *concept assertion*. Assume that $\mathcal{I} \models \langle a : C \bowtie \gamma \rangle$. We show, by induction on the structure of concepts and roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \bowtie \gamma \rangle)$.
   - *Atomic concept*. Assume that $\mathcal{I} \models \langle a : A \geqslant \alpha \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant \alpha$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_c} : A^{\mathcal{I}_c}_{\geqslant \alpha}$. Consequently, $a^{\mathcal{I}_c} \in (\rho(A, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \geqslant \alpha \rangle)$. Now assume that $\mathcal{I} \models \langle a : A \leqslant \beta \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant \beta$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_c} \notin A^{\mathcal{I}_c}_{>\beta} \iff a^{\mathcal{I}_c} \in \neg A^{\mathcal{I}_c}_{>\beta} \iff a^{\mathcal{I}_c} \in (\rho(A, \leqslant \beta))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \leqslant \beta) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \leqslant \beta \rangle)$.
   - *Top concept*. Assume that $\mathcal{I} \models \langle a : \top \geqslant \alpha \rangle$. Then, $\top^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant \alpha$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_c} \in \Delta^{\mathcal{I}_c} = \top$. Consequently, $a^{\mathcal{I}_c} \in (\rho(\top, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\top, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \top \geqslant \alpha \rangle)$. The case $\mathcal{I} \models \langle a : \top \leqslant \beta \rangle$ is not possible. If $\mathcal{I} \models \langle a : \top \leqslant \beta \rangle$ we have that $1 \leqslant \beta$, which is a contradiction with the restriction $\beta \in [0,1)$. If $\mathcal{I} \models \langle a : \top < \alpha \rangle$ we have that $1 < \alpha$, which is a contradiction with the restriction $\alpha \in (0,1]$.
   - *Bottom concept*. This case is similar to the previous one.
   - *Concept negation*. Assume that $\mathcal{I} \models \langle a : \neg C \bowtie \gamma \rangle$. Then, $1 - C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie^{-} 1 - \gamma$. By induction hypothesis, $a^{\mathcal{I}_c} \in (\rho(C, \bowtie^{-} 1 - \gamma))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a \in \rho(C, \bowtie^{-} 1 - \gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \neg C \bowtie \gamma \rangle)$.
   - *Concept conjunction*. Assume that $\mathcal{I} \models \langle a : C \sqcap D \geqslant \alpha \rangle$. Then, $\max\{C^{\mathcal{I}}(a^{\mathcal{I}}) + D^{\mathcal{I}}(a^{\mathcal{I}}) - 1, 0\} \geqslant \alpha$. Since $\alpha \in (0,1]$, it follows that $C^{\mathcal{I}}(a) + D^{\mathcal{I}}(a) \geqslant 1 + \alpha$. We do not know exactly the degrees of truth of $C^{\mathcal{I}}(a)$ and $D^{\mathcal{I}}(a)$, but using Proposition 4.1 we know that they certainly belong to $\mathcal{N}$, so $C^{\mathcal{I}}(a) \geqslant \gamma_1$ and $D^{\mathcal{I}}(a) \geqslant \gamma_2$, for some $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 + \gamma_2 \geqslant 1 + \alpha$. By induction hypothesis, $a^{\mathcal{I}_c} \in (\rho(C, \geqslant \gamma_1))^{\mathcal{I}_c} \cap (\rho(D, \geqslant \gamma_2))^{\mathcal{I}_c}$.
     Now, it is possible to build a disjunction over all the pairs of individuals $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 + \gamma_2 \geqslant 1 + \alpha$, and $a^{\mathcal{I}_c} \in (\bigsqcup_{\gamma_1,\gamma_2}\{\rho(C, \geqslant \alpha_1) \sqcap \rho(D, \geqslant \gamma_2)\})^{\mathcal{I}_c}$. By Proposition 4.3, this can simplified to the degrees $\gamma_1 + \gamma_2 = 1 + \alpha$. This is equivalent to $a^{\mathcal{I}_c} \in (\rho(C \sqcap D, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C \sqcap D, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcap D \geqslant \alpha \rangle)$.
     In the case $\mathcal{I} \models \langle a : C \sqcap D \leqslant \beta \rangle$, we use the equivalence $C \sqcap D \equiv \neg(\neg C \sqcup \neg D)$ and consider $\mathcal{I} \models \langle a : \neg C \sqcup \neg D \geqslant 1 - \beta \rangle$.
   - *Concept disjunction*. This case is similar to concept conjunction. Assume that $\mathcal{I} \models \langle a : C \sqcup D \geqslant \alpha \rangle$. Then, $\min\{C^{\mathcal{I}}(a^{\mathcal{I}}) + D^{\mathcal{I}}(a^{\mathcal{I}}), 1\} \geqslant \alpha$. It follows that $C^{\mathcal{I}}(a) + D^{\mathcal{I}}(a) \geqslant \alpha$. Now, there are three possibilities:
     (a) $C^{\mathcal{I}}(a) \geqslant \alpha$, or
     (b) $D^{\mathcal{I}}(a) \geqslant \alpha$, or
     (c) $C^{\mathcal{I}}(a) \geqslant \gamma_1, D^{\mathcal{I}}(a) \geqslant \gamma_2$, for some $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 + \gamma_2 \geqslant \alpha$. Using Proposition 4.3, this can simplified to the degrees $\gamma_1 + \gamma_2 = \alpha$.

     By induction hypothesis, $a^{\mathcal{I}_c} \in (\rho(C, \geqslant \alpha) \sqcup \rho(D, \geqslant \alpha) \bigsqcup_{\gamma_1,\gamma_2}\{\rho(C, \geqslant \gamma_1) \sqcap \rho(D, \geqslant \gamma_2)\})^{\mathcal{I}_c}$, for every $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 + \gamma_2 \geqslant \alpha$. This is equivalent to say that $a^{\mathcal{I}_c} \in (\rho(C \sqcup D, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcup D \geqslant \alpha \rangle)$.
     In the case $\mathcal{I} \models \langle a : C \sqcup D \leqslant \beta \rangle$, we use the equivalence $C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$ and consider $\mathcal{I} \models \langle a : \neg C \sqcap \neg D \geqslant 1 - \beta \rangle$.

   - *Universal quantification*. Assume that $\mathcal{I} \models \langle a : \forall R.C \geqslant \alpha \rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow D^{\mathcal{I}}(b)\} \geqslant \alpha$. Hence, for an arbitrary individual $b \in \Delta^{\mathcal{I}}$ it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow C^{\mathcal{I}}(b) \geqslant \alpha$. Now, one of the following conditions holds:
     (a) $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leqslant C^{\mathcal{I}}(b)$ (which makes the Łukasiewicz implication equal to $1 \geqslant \alpha$), or
     (b) $1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b) + C^{\mathcal{I}}(b) \geqslant \alpha$ (which makes the implication take a value $\geqslant \alpha$).

     Note that condition *(b)* is equivalent to $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leqslant C^{\mathcal{I}}(b) + 1 - \alpha$, which subsumes condition *(a)* since $\alpha \in (0,1]$. Now, whatever γ_1 is, $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geqslant \gamma_1$ implies that $C^{\mathcal{I}}(b) \geqslant (\gamma_1 - (1 - \alpha))$. Equivalently (and using Proposition 4.3), given $\gamma_1 = \gamma_2 + 1 - \alpha$, $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geqslant \gamma_1$ implies that $C^{\mathcal{I}}(b) \geqslant \gamma_2$.
     By induction hypothesis, $(a^{\mathcal{I}_c}, b) \in (\rho(R, \geqslant \gamma_1))^{\mathcal{I}_c}$ implies $b \in (\rho(C, \geqslant \gamma_2))^{\mathcal{I}_c}$ for every $\gamma_1 = \gamma_2 + 1 - \alpha$. This is exactly the semantics of $a^{\mathcal{I}_c} \in (\sqcap_{\gamma_1,\gamma_2}\{\forall \rho(R, \geqslant \gamma_1).\rho(C, \geqslant \gamma_2)\})^{\mathcal{I}_c}$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$ and $\gamma_1 = \gamma_2 + 1 - \alpha$. This is equivalent to $a^{\mathcal{I}_c} \in (\rho(\forall R.C, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\forall R.C, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \forall R.C \geqslant \alpha \rangle)$.
     In the case $\mathcal{I} \models \langle a : \forall R.C \leqslant \beta \rangle$ we use the equivalence $\forall R.C \equiv \neg(\exists R.(\neg C))$ and consider $\mathcal{I} \models \langle a : \exists R.(\neg C) \geqslant 1 - \beta \rangle$.

   - *Existential quantification*. This case is similar to universal quantification. Assume that $\mathcal{I} \models \langle a : \exists R.C \geqslant \alpha \rangle$. Then, $\sup_{b \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a^{\mathcal{I}}, b) \otimes C^{\mathcal{I}}(b)\} \geqslant \alpha$. Due to the WMP, there exists an individual $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \otimes C^{\mathcal{I}}(b)) \geqslant \alpha \iff R^{\mathcal{I}}(a^{\mathcal{I}}, b) + C^{\mathcal{I}}(b) \geqslant 1 + \alpha \iff R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geqslant \gamma_1$ and $C^{\mathcal{I}}(b) \geqslant \gamma_2$ for some $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 + \gamma_2 \geqslant 1 - \alpha$. Equivalently (and using Proposition 4.3), this can be simplified to $\gamma_1 + \gamma_2 = 1 - \alpha$.
     By induction hypothesis, for some $b \in \Delta^{\mathcal{I}_c}$, $(a^{\mathcal{I}_c}, b) \in (\rho(R, \geqslant \gamma_1))^{\mathcal{I}_c}$ and $b \in (\rho(C, \geqslant \gamma_2))^{\mathcal{I}_c}$ for some $\gamma_1 + \gamma_2 = 1 - \alpha$. This is exactly the semantics of $a^{\mathcal{I}_c} \in (\bigsqcup_{\gamma_1,\gamma_2}\{\exists \rho(R, \geqslant \gamma_1).\rho(C, \geqslant \gamma_2)\})^{\mathcal{I}_c}$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$ and $\gamma_1 = \gamma_2 + 1 - \alpha$. This is equivalent to say that $a^{\mathcal{I}_c} \in (\rho(\exists R.C, \geqslant \alpha))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\exists R.C, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \exists R.C \geqslant \alpha \rangle)$.
     In the case $\mathcal{I} \models \langle a : \exists R.C \leqslant \beta \rangle$ we use the equivalence $\exists R.C \equiv \neg(\forall R.(\neg C))$ and consider $\mathcal{I} \models \langle a : \forall R.(\neg C) \geqslant 1 - \beta \rangle$.
   - *Fuzzy nominals*. Assume that $\mathcal{I} \models \langle a : \{\alpha_1/o_1, \ldots, \alpha_n/o_n\} \geqslant \alpha \rangle$. Let $o_{i1}, \ldots, o_{ik}$ be such that $\alpha_{ij} \geqslant \alpha$. Then, $\sup\{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \geqslant \alpha$, with $a^{\mathcal{I}} \in \{o_{i_1}, \ldots, o_{i_k}\}^{\mathcal{I}}$. By construction of $\mathcal{I}_{\mathcal{C}}$, it holds that $a^{\mathcal{I}_c} \in \{o_{i_1}, \ldots, o_{i_k}\}^{\mathcal{I}_c} \iff$

$a^{\mathcal{I}_C} \in (\rho(\{\alpha_1/o_1,\ldots,\alpha_n/o_n\},\geqslant\alpha))^{\mathcal{I}_C} \iff \mathcal{I}_C \models a:\rho(\{\alpha_1/o_1,\ldots,\alpha_n/o_n\},\geqslant\alpha))\rangle \iff \mathcal{I}_C \models \kappa(\langle a:\{\alpha_1/o_1,\ldots,\alpha_n/o_n\}\geqslant\alpha\rangle)$. The case $\leqslant\beta$ is quite straightforward.

- *At-least qualified number restriction*. Assume $\mathcal{I} \models \langle a:(\geqslant mS.C)\geqslant\alpha\rangle$. Then, $\sup_{b_1,\ldots,b_m\in\Delta^{\mathcal{I}}}\{\min_{i=1}^m\{S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)\}\otimes(\otimes_{1\leqslant j<k\leqslant m}\{b_j\neq b_k\})\}\geqslant\alpha$. Note that $(\otimes_{1\leqslant j<k\leqslant m}\{b_j\neq b_k\})$ can be either 0 or 1. If it is 0, then we have that $\sup_{b_1,\ldots,b_m\in\Delta^{\mathcal{I}}}\{\min_{i=1}^m\{S^{\mathcal{I}}(a^{\mathcal{I}},b_i\otimes C^{\mathcal{I}}(b_i)\}\}\otimes 0\}=0\geqslant\alpha$, which is not possible because by definition $\alpha\in(0,1]$. Hence, $(\otimes_{1\leqslant j<k\leqslant m}\{b_j\neq b_k\})=1$ (which means that the elements are pairwise different).
  Thanks to the WMP, for some pairwise different elements $b_1,\ldots,b_m\in\Delta^{\mathcal{I}}$, $\min_{i=1}^m\{S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)\}\otimes 1 = \min_{i=1}^m\{S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)\}\geqslant\alpha$. This means that every b_i satisfies $S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)\}\geqslant\alpha$. Similarly as with the conjunction, this means that $S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\geqslant\gamma_i^1$ and $C^{\mathcal{I}}(b_i)\geqslant\gamma_i^2$, for some $\gamma_i^1,\gamma_i^2\in\mathcal{N}^+$ such that $\gamma_i^1+\gamma_i^2\geqslant 1+\alpha$. Furthermore, thanks to Proposition 4.6, we know that $b_i\in B_i^{\mathcal{I}}$, for some new concepts $B_1,\ldots,B_m$ forming a partition w.r.t. $\mathcal{I}$.[5]
  By induction hypothesis, there exist individuals $b_1,\ldots,b_m\in\Delta^{\mathcal{I}_C}$ such that $(a^{\mathcal{I}_C},b_i)\in(\rho(S,\geqslant\gamma_i^1))^{\mathcal{I}_C}$ and $b_i\in((\rho(C,\geqslant\gamma_i^2))^{\mathcal{I}_C}\cap(B_i)^{\mathcal{I}_C})$, for some $\gamma_i^1,\gamma_i^2\in\mathcal{N}^+$ such that $\gamma_i^1+\gamma_i^2\geqslant 1+\alpha$. Clearly, B_i also form a partition w.r.t. $\mathcal{I}_c$. This is exactly the semantics of $a^{\mathcal{I}_C}\in\bigsqcup_{\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2}\{\exists\rho(S,\geqslant\gamma_1^1).(B_1\sqcap\rho(C,\geqslant\gamma_1^2))\sqcap\ldots\sqcap\exists\rho(S,\geqslant\gamma_m^1).(B_m\sqcap\rho(C,\geqslant\gamma_m^2))\}$, for every combination of $\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2\in\mathcal{N}^+$ such that $\gamma_i^1+\gamma_i^2=1+\alpha$, for $i=\{1,\ldots,m\}$. This is equivalent to $a^{\mathcal{I}_C}\in(\rho((\geqslant mS.C),\geqslant\alpha))^{\mathcal{I}_C}\iff\mathcal{I}_C\models a:\rho((\geqslant mS.C),\geqslant\alpha)\iff\mathcal{I}_C\models\kappa(\langle a:(\geqslant mS.C)\geqslant\alpha\rangle)$.
  In the case $\mathcal{I}\models\langle a:(\geqslant mS.C)\leqslant\beta\rangle$ we have that either $(\otimes_{1\leqslant j<k\leqslant m}\{b_j\neq b_k\})=0$ or $\min_{i=1}^m\{S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)\}\leqslant\beta$. In other words, there does not exist m pairwise different elements such that $S^{\mathcal{I}}(a^{\mathcal{I}},b_i)\otimes C^{\mathcal{I}}(b_i)>\beta$. Now, we consider all the possible pairs of minimal degrees $\gamma_i^1,\gamma_i^2\in\mathcal{N}^+$ such that $\gamma_i^1+\gamma_i^2>1+\beta$ and obtain that $a^{\mathcal{I}_C}\in\neg\Big(\Big[\bigsqcup_{\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2}\{\exists\rho(S,\geqslant\gamma_1^1).(B_1\sqcap\rho(C,\geqslant\gamma_1^2))\sqcap\cdots\sqcap\exists\rho(S,\geqslant\gamma_m^1).(B_m\sqcap\rho(C,\geqslant\gamma_m^2))\}\Big]\Big)^{\mathcal{I}_C}$, for every combination of $\gamma_1^1,\gamma_1^2,\ldots,\gamma_m^1,\gamma_m^2\in\mathcal{N}^+$ such that *(i)* $\gamma_i^1+\gamma_i^2>1+\beta$, for $i=\{1,\ldots,m\}$, and *(ii)* $\nexists\gamma\in\mathcal{N}^+$ such that $\gamma<\gamma_i^1$ and $\gamma+\gamma_i^2>1+\beta$, and *(iii)* $\nexists\gamma\in\mathcal{N}^+$ such that $\gamma<\gamma_i^2$ and $\gamma_i^1+\gamma>1+\beta$. This is equivalent to $a^{\mathcal{I}_C}\in(\rho((\geqslant mS.C),\leqslant\beta))^{\mathcal{I}_C}\iff\mathcal{I}_C\models a:\rho((\geqslant mS.C),\leqslant\beta)\iff\mathcal{I}_C\models\kappa(\langle a:(\geqslant mS.C)\leqslant\beta\rangle)$.
- *At-most qualified number restriction*. In the case $\mathcal{I}\models\langle a:(\leqslant nS.C)\bowtie\gamma\rangle$ we use the equivalence $\leqslant nS.C\equiv\neg(\geqslant n+1S.C)$ and consider $\mathcal{I}\models\langle a:\neg(\geqslant n+1S.C)\bowtie\gamma\rangle$.
- *Local reflexivity*. Assume that $\mathcal{I}\models\langle a:\exists S.\texttt{Self}\geqslant\alpha\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}},a^{\mathcal{I}})\geqslant\alpha$. By induction hypothesis, $(a^{\mathcal{I}_C},a^{\mathcal{I}_C})\in(\rho(S,\geqslant\alpha))^{\mathcal{I}_C}\iff\mathcal{I}_C\models(a,a):\rho(S,\geqslant\alpha)\iff\mathcal{I}_C\models\kappa(\langle a:\exists S.\texttt{Self}\geqslant\alpha\rangle)$. Now assume that $\mathcal{I}\models\langle a:\exists S.\texttt{Self}\leqslant\beta\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}},a^{\mathcal{I}})\leqslant\beta$. By induction hypothesis, $(a^{\mathcal{I}_C},a^{\mathcal{I}_C})\in(\rho(S,\leqslant\beta))^{\mathcal{I}_C}$. Hence, it follows that $(a^{\mathcal{I}_C},a^{\mathcal{I}_C})\notin(\rho(S,\neg\leqslant\beta))^{\mathcal{I}_C}\iff(a^{\mathcal{I}_C},a^{\mathcal{I}_C})\in\neg(\rho(S,\neg\leqslant\beta))^{\mathcal{I}_C}\iff a^{\mathcal{I}_C}\in(\rho(\exists S.\texttt{Self},\leqslant\beta))^{\mathcal{I}_C}\iff\mathcal{I}_C\models a:\rho(\exists S.\texttt{Self},\leqslant\beta)\iff\mathcal{I}_C\models\kappa(\langle a:\exists S.\texttt{Self},\leqslant\beta\rangle)$.

5. τ is a *fuzzy GCI*. Assume that $\mathcal{I}\models\langle C\sqsubseteq D\geqslant\alpha\rangle$. Then, $\inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\}\geqslant\alpha$. Hence, for an arbitrary individual $x\in\Delta^{\mathcal{I}}$ it follows that $C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\geqslant\alpha$ and hence one of the following conditions holds:
   (a) $C^{\mathcal{I}}(x)\leqslant D^{\mathcal{I}}(x)$ (which makes the Łukasiewicz implication equal to $1\geqslant\alpha$), or
   (b) $1-C^{\mathcal{I}}(x)+D^{\mathcal{I}}(x)\geqslant\alpha$ (which makes the implication take a value $\geqslant\alpha$).

   Note that condition *(b)* is equivalent to $C^{\mathcal{I}}(x)\leqslant D^{\mathcal{I}}(x)+1-\alpha$, which subsumes condition *(a)* since $\alpha\in(0,1]$. Now, whatever γ_1 is, $\rho(C,\geqslant\gamma_1)$ implies that $D^{\mathcal{I}}(x)\geqslant(\gamma_1-(1-\alpha))$. Equivalently, given $\gamma_1=\gamma_2+1-\alpha$, $\rho(C,\geqslant\gamma_1)$ implies $\rho(D,\geqslant\gamma_2)$. This is exactly the semantics of $\bigcup_{\gamma_1,\gamma_2}\{\rho(C,\geqslant\gamma_1)\sqsubseteq\rho(D,\geqslant\gamma_2)\}$ for every pair $\langle\gamma_1,\gamma_2\rangle$ such that $\gamma_1,\gamma_2\in\mathcal{N}^+$ and $\gamma_1=\gamma_2+1-\alpha$.

6. τ is a *fuzzy RIA*. Assume that $\mathcal{I}\models\langle R_1\ldots R_m\sqsubseteq R\geqslant\alpha\rangle$. The case is similar to the previous one, but now for an arbitrary pair of individuals $x_1,x_{n+1}\in\Delta^{\mathcal{I}}$ it follows that $\sup_{x_2\ldots x_n\in\Delta^{\mathcal{I}}}\{R_1^{\mathcal{I}}(x_1,x_2)\otimes\cdots\otimes R_n^{\mathcal{I}}(x_n,x_{n+1})\}\Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})\geqslant\alpha$. Consequently, $\sup_{x_2\ldots x_n\in\Delta^{\mathcal{I}}}\{R_1^{\mathcal{I}}(x_1,x_2)\otimes\cdots\otimes R_n^{\mathcal{I}}(x_n,x_{n+1})\}\leqslant R^{\mathcal{I}}(x_1,x_{n+1})+1-\alpha$.
   Using the WMP, for some $x_2\ldots x_n\in\Delta^{\mathcal{I}}$ (the elements in the supremum), $\max\{R_1^{\mathcal{I}}(x_1,x_2)+\cdots+R_n^{\mathcal{I}}(x_n,x_{n+1})-(n-1),0\}\leqslant R^{\mathcal{I}}(x_1,x_{n+1})+1-\alpha$. The case where the left side is 0 is not interesting, since $0\leqslant R^{\mathcal{I}}(x_1,x_{n+1})+1-\alpha$ is a tautology. So, we focus on the other case, and it follows that $R_1^{\mathcal{I}}(x_1,x_2)+\cdots+R_n^{\mathcal{I}}(x_n,x_{n+1})-(n-1)\leqslant R^{\mathcal{I}}(x_1,x_{n+1})+1-\alpha\iff R_1^{\mathcal{I}}(x_1,x_2)+\cdots+R_n^{\mathcal{I}}(x_n,x_{n+1})\leqslant R^{\mathcal{I}}(x_1,x_{n+1})+n-\alpha$.
   Now, the case is similar as before for the elements in the supremum $x_2\ldots x_n$. Finally, we obtain $\bigcup_{\gamma_1,\ldots,\gamma_{n+1}}\{\rho(R_1,\geqslant\gamma_1)\ldots\rho(R_n,\geqslant\gamma_n)\sqsubseteq\rho(R,\geqslant\gamma_{n+1})\}$ for every combination of elements $\langle\gamma_1,\ldots,\gamma_{n+1}\rangle$ such that $\gamma_1,\ldots,\gamma_{n+1}\in\mathcal{N}^+$ and $\gamma_1+\cdots+\gamma_n=\gamma_{n+1}+n-\alpha$.
7. τ is a *role disjoint* axiom. Assume that $\mathcal{I}\models\texttt{dis}(S_1,S_2)$. Then, $\forall x,y\in\Delta^{\mathcal{I}},S_1^{\mathcal{I}}(x,y)=0$ or $S_2^{\mathcal{I}}(x,y)=0$. By induction, $\forall x,y\in\Delta^{\mathcal{I}_C},(x,y)\in(\rho(S_1,\leqslant 0))^{\mathcal{I}_C}$ or $(x,y)\in(\rho(S_2,\leqslant 0))^{\mathcal{I}_C}\iff\forall x,y\in\Delta^{\mathcal{I}_C},(x,y)\notin(\rho(S_1,>0))^{\mathcal{I}_C}$ or $(x,y)\notin(\rho(S_2,>0))^{\mathcal{I}_C}\iff(\rho(S_1,>0))^{\mathcal{I}_C}\cap(\rho(S_2,>0))^{\mathcal{I}_C}=\emptyset\iff\mathcal{I}_C\models(dis(\rho(S_1,>0),\rho(S_2,>0)))\iff\mathcal{I}_C\models\kappa(\texttt{dis}(S_1,S_2))$.
8. τ is a *reflexive* role axiom. Assume that $\mathcal{I}\models\texttt{ref}(R)$. Then, $\forall x\in\Delta^{\mathcal{I}},R^{\mathcal{I}}(x,x)=1$. By induction, $\forall x\in\Delta^{\mathcal{I}_C}$, $(x,x)\in(\rho(R,\geqslant 1))^{\mathcal{I}_C}\iff\forall x\in\Delta^{\mathcal{I}_C},\mathcal{I}_C\models(x,x):\rho(R,\geqslant 1)\iff\mathcal{I}_C\models\kappa(\texttt{ref}(R))$.
9. τ is an irreflexive role axiom. Assume that $\mathcal{I}\models\texttt{irr}(S)$. Then, $\forall x\in\Delta^{\mathcal{I}},S^{\mathcal{I}}(x,x)=0$. By induction, $\forall x\in\Delta^{\mathcal{I}_C},(x,x)\in(\rho(S,\leqslant 0))^{\mathcal{I}_C}\iff\forall x\in\Delta^{\mathcal{I}_C},(x,x)\notin(\rho(S,>0))^{\mathcal{I}_C}\iff\mathcal{I}_C\models\texttt{irr}(\rho(S,>0))\iff\mathcal{I}_C\models\kappa(\texttt{irr}(S))$.

[5] Obviously, it is mandatory to add the axioms $\top\sqsubseteq B_1\sqcup B_2\sqcup\cdots\sqcup B_m$ and $B_i\sqcap B_j\sqsubseteq\bot$, for $i<j$, in order to guarantee that they indeed form a partition.

10. τ is an *asymmetry* role axiom. Assume that $\mathcal{I} \models \mathtt{asy}(S)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$. By induction, $\forall x, y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S, > 0))^{\mathcal{I}_C}$ then $(y,x) \in (\rho(S, \leqslant 0))^{\mathcal{I}_C} \iff \forall x, y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S, > 0))^{\mathcal{I}_C}$ then $(y,x) \notin (\rho(S, > 0))^{\mathcal{I}_C}$. Consequently, $\mathcal{I}_{\mathcal{C}} \models \kappa(\mathtt{asy}(\rho(S, > 0)))$.

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If $\mathsf{crisp}(\mathcal{K})$ is satisfiable, it is not possible (due to the axioms in $T(N^{\mathcal{K}})$) to have an individual a such that $a^{\mathcal{I}_C} \in (A_{\triangleright\gamma_1})^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \notin (A_{\triangleright\gamma_2})^{\mathcal{I}_C}$ with $\gamma_2 < \gamma_1$, so for every individual a we can compute the maximum value α such that $a{:}A_{\geqslant\alpha}$ holds, or the maximum value β such that $a{:}A_{>\beta}$ holds, and use these values in the construction of the fuzzy interpretation. The case for roles in $R(N^{\mathcal{K}})$ is similar. □

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