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JOINING GÖDEL AND ZADEH FUZZY LOGICS IN FUZZY DESCRIPTION LOGICS

FERNANDO BOBILLO

*Department of Computer Science and Systems Engineering,
University of Zaragoza, Spain
fbobillo@unizar.es*

MIGUEL DELGADO

*Department of Computer Science and Artificial Intelligence,
University of Granada, Spain
mdelgado@ugr.es*

JUAN GÓMEZ-ROMERO

*Applied Artificial Intelligence Group,
University Carlos III, Madrid, Spain
jgromero@inf.uc3m.es*

UMBERTO STRACCIA

*Istituto di Scienza e Tecnologie dell'Informazione (ISTI-CNR), Pisa, Italy
straccia@isti.cnr.it*

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Ontologies have succeeded as a knowledge representation formalism in many domains of application. Nevertheless, they are not suitable to represent vague or imprecise information. To overcome this limitation, several extensions to classical ontologies based on fuzzy logic have been proposed. Even though different fuzzy logics lead to fuzzy ontologies with very different logical properties, the combined use of different fuzzy logics has received little attention to date. This paper proposes a fuzzy extension of the Description Logic $\mathcal{SROIQ}$ — the logic behind the ontology language OWL 2 — that joins Gödel and Zadeh fuzzy logics. We analyze the properties of the new fuzzy Description Logic in order to provide guidelines to ontology developers to exploit the best features of each fuzzy logic. The proposal also considers degrees of truth belonging to a finite set of linguistic terms rather than numerical values, thus being closer to real experts' reasonings. We prove the decidability of the combined logic by presenting a reasoning preserving procedure to obtain a crisp representation for it. This result is generalized to offer a similar reduction that can be applied when any other finite t-norms, t-conorms, negations or implications are considered in the logic.

Keywords: Fuzzy description logics; fuzzy ontologies; logic for the semantic web.

1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web — an extension to the current web to give information a well-defined meaning.[1] An ontology is defined as an explicit and formal specification of a shared conceptualization.[2] This definition states that ontologies represent the entities (concepts, relationships, individuals) in a domain promoting interrelation with other models and automatic processing. Ontologies allow adding semantics to data, thus making knowledge maintenance, information integration, and component reuse easier. The language OWL 2[3] has recently become a W3C Recommendation for ontology representation.

Description Logics (DLs)[4] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to be very useful as ontology languages.[5] In fact, OWL 2 is almost equivalent to $\mathcal{SROIQ}(\mathbf{D})$.

Nevertheless, classical ontologies based on DLs are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real world domains.[7] Since fuzzy set theory and fuzzy logic are suitable formalisms to handle these types of knowledge, several fuzzy extensions of DLs have been proposed,[9] yielding fuzzy ontologies. Fuzzy ontologies have proved to be useful in several applications, such as information retrieval,[10–12] image interpretation,[13–15] the Semantic Web and the Internet,[7,16,17] and economy,[8] among many others.[18–21]

There are three main fuzzy logics: Łukasiewicz, Gödel, and Product. The importance of these three fuzzy logics is due to the fact that any continuous t-norm can be obtained as a combination of Łukasiewicz, Gödel, and Product t-norm.[22] In addition to these fuzzy logics, it is also common to distinguish the Zadeh fuzzy logic, which includes the fuzzy connectives considered in his seminal work; namely Gödel conjunction and disjunction, Łukasiewicz negation and Kleene-Dienes implication.

Different families of fuzzy operators (or fuzzy logics) lead to fuzzy DLs with different properties. For example, Gödel and Zadeh fuzzy logics include an idempotent conjunction (minimum); e.g., $\min\{0.5, 0.5, 0.5\} = \min\{0.5, 0.5\} = 0.5$. This property implies that some inferences in the corresponding fuzzy DL are independent of the granularity of the fuzzy ontology. This is a desired property most times (e.g., Ref. 11), but not always.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory,[9] existing works are limited to one fuzzy logic — mostly Zadeh's. Unfortunately, the combination of different fuzzy logics in the setting of fuzzy DLs has received little attention, even though it would allow fuzzy ontology developers to freely choose the logic with the logical properties that best suit to a particular application domain.

Let us illustrate why combining two fuzzy logics is of interest by considering a concrete example: the cases of Gödel and Zadeh fuzzy logics. On the one hand, Gödel fuzzy DLs have some usually undesired logical properties. For instance, they have a not involutive negation. If we define the fuzzy concept `DarkThing` as the negation of the fuzzy concept `LightThing`, it follows that $\neg\texttt{DarkThing} \neq \texttt{LightThing}$. Moreover, if ≤ 1 `hasParent.Blond` denotes the fuzzy set of people having no more than one blond parent and ≥ 2 `hasParent.Blond` denotes the fuzzy set of people with both a blond parent and a blond mother, then it follows that $\geq 2\ \texttt{hasParent.Blond} \neq \neg(\leq 1\ \texttt{hasParent.Blond})$. On the other hand, although these problems do not happen in Zadeh fuzzy DLs, these logics also lead to some unnatural inferences. If we query to a fuzzy ontology reasoner whether a concept is more general than itself, the answer is not always true as it should. Nevertheless, in Gödel fuzzy DLs, the answer would be correct.

In this paper, we define a fuzzy extension of the DL $\mathcal{SROIQ}$ that joins Gödel and Zadeh fuzzy logics (called $GZ\ \mathcal{SROIQ}$) in order to combine the advantages of both of them. We discuss the properties of $GZ\ \mathcal{SROIQ}$ and present a reasoning procedure for the finite-valued case. The underlying fuzzy logic is the finite Gödel logic G_n extended with an involutive negation (thus making Kleene-Dienes implication definable).

In fact, we assume a finite (totally ordered) set of linguistic terms or labels; e.g., $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$, rather than dealing with degrees of truth in $[0,1]$ as usually in fuzzy DLs. This modification is very useful, since experts tend to reason by relying on a set of linguistic terms, and the numerical interpretations of these labels can be avoided.[27,28] Last but not least, finite fuzzy DLs are becoming more important nowadays because in $[0,1]$ fuzzy DLs with a t-norm containing Łukasiewicz or Product operators may cause undecidability even for not very expressive logics.[29–32] Although the use of linguistic labels as degrees in fuzzy DLs has previously been proposed,[33] our approach is a step further in this direction, because we also extend the notion of fuzzy interpretations to the case of a finite chain. This conceptually small modification results in important differences in the reasoning algorithms and consequently there is an outstanding improvement in their performances.

$GZ\ \mathcal{SROIQ}$ is the logic accepted by the fuzzy ontology reasoner DeLorean,[24] which to date is the only fuzzy ontology reasoner that supports a fuzzy extension of OWL 2.[25,a] An additional advantage, from a theoretical point of view, is that the increase of the computational complexity of the fuzzy DL is not significantly higher. Furthermore, our Zadeh Gödel fuzzy DL is closely related to the logic $\mathcal{ALC}^G(\mathbf{S})$. For every continuous t-norm $*$, $\mathcal{ALC}^*(\mathbf{S})$ is the fuzzy DL that corresponds to the first order t-norm based logic $L^*_\sim(\mathbf{S})\forall$.[26] Therefore, it is possible to reuse well-known results from Mathematical Fuzzy Logic in the study of this fuzzy DL.

[a]The reasoner fuzzyDL supports a significant fraction but not fuzzy OWL 2 as a whole.[34]

The remainder of this paper is organized as follows. The following section recalls some preliminaries on mathematical finite fuzzy logics. Then, Sec. 3 defines a fuzzy extension of the very expressive DL $\mathcal{SROIQ}$ based on Gödel and Zadeh fuzzy logics and discusses some logical properties. Section 4 show the decidability of the logic by providing a reduction of fuzzy $\mathcal{SROIQ}$ with Gödel and Zadeh connectives into crisp $\mathcal{SROIQ}$. Finally, Sec. 5 reviews some related work and Sec. 6 sets out some conclusions and ideas for future research.

2. Finite Fuzzy Logics

Fuzzy set theory and fuzzy logic were proposed by L. Zadeh[37] to manage imprecise and vague knowledge. Let X be a set of elements called the reference set. A *fuzzy subset* A of X is defined by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns any $x \in X$ to a value in the interval of real numbers between 0 and 1. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which x can be considered as an element of X. All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication set operations are performed by a t-norm function, a t-conorm function, a negation function and an implication function, respectively.

From now on, we will consider finite chains of degrees of truth.[27,28] For our purposes, all finite chains with the same number of elements are equivalent. Therefore, we will deal with the simplest finite chain of $p + 1$ elements: $\mathcal{N} = \{0 = \gamma_0 < \gamma_1 < \ldots < \gamma_p = 1\}$, where $p \geq 1$. Note that when $p = 1$ we have a classical two-valued logic. Such a $\mathcal{N}$ can be understood as a set of linguistic terms or labels. For example, $\{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$. From a practical point of view, a small p is sufficient in many applications.

T-norms, t-conorms, negations and implications can be restricted to finite chains. A t-norm $\otimes$ is a binary operation on $\mathcal{N}$ which is commutative, associative, monotone and has γ_p as neutral element. A t-conorm $\oplus$ is a binary operation on $\mathcal{N}$ which is commutative, associative, monotone and has γ_0 as neutral element. A negation $\ominus$ is a unary operation on $\mathcal{N}$ which is monotone and verifies $\ominus\gamma_0 = \gamma_p, \ominus\gamma_p = \gamma_0$. An *involutive* negation satisfies that $\ominus(\ominus\gamma) = \gamma$. There is only one involutive negation over a finite chain $\mathcal{N}$, and is defined as $\ominus_Z\gamma_i = \gamma_{p-i}$. For every $\gamma_i \in \mathcal{N}$, $\ominus_Z\gamma_i$ denotes its complementary degree. An implication $\Rightarrow$ is a binary operation on $\mathcal{N}$ which is non-increasing in the first argument, non-decreasing in the second argument, and verifies the boundary conditions $(\gamma_0 \Rightarrow \gamma_0) = (\gamma_p \Rightarrow \gamma_p) = \gamma_p$ and $(\gamma_p \Rightarrow \gamma_0) = \gamma_0$.

Table 1 shows the most important fuzzy logics under finite chains: Zadeh, Łukasiewicz, and Gödel. A product-based fuzzy logic cannot be defined over a finite chain since, in general, $\gamma_i \otimes \gamma_j \notin \mathcal{N}$. Besides, the implication of the Zadeh fuzzy logic (which is called Kleene-Dienes) and Łukasiewicz implication are S-implications.

For every $\gamma \in \mathcal{N}$, the γ-*cut* of a fuzzy set A is defined as the (crisp) set such that its elements belong to A with degree at least γ, i.e. $\{x \mid \mu_A(x) \geq \gamma\}$.

Table 1. Popular fuzzy logics over a finite chain.

Family	$\gamma_i \otimes \gamma_j$	$\gamma_i \oplus \gamma_j$	$\ominus \gamma_i$	$\gamma_i \Rightarrow \gamma_j$
Zadeh	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	γ_{p-i}	$\max\{\gamma_{p-i}, \gamma_j\}$
Gödel	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	$\begin{cases} \gamma_p, & \text{if } \gamma_i = 0 \\ \gamma_0, & \text{if } \gamma_i > 0 \end{cases}$	$\begin{cases} \gamma_p, & \text{if } \gamma_i \leq \gamma_j \\ \gamma_j, & \text{if } \gamma_i > \gamma_j \end{cases}$
Łukasiewicz	$\gamma_{\max\{i+j-p,0\}}$	$\gamma_{\min\{i+j,p\}}$	γ_{p-i}	$\gamma_{\min\{p-i+j,p\}}$

In the fuzzy DL that we consider in this paper, *fuzzy statements* have the form $\phi \geq \gamma$ and encode that the degree of truth of ϕ is *at least* γ, being $\gamma \in \mathcal{N}$ and ϕ a statement. Respectively, $\phi \leq \gamma$ represents that the degree of truth of ϕ is *at most* γ.

A fuzzy model $\mathcal{I}$ *satisfies* a fuzzy statement $\phi \geq \gamma$ or $\mathcal{I}$ is a *model* of $\phi \geq \gamma$, denoted $\mathcal{I} \models \phi \geq \gamma$, iff $\mathcal{I}(\phi) \geq \gamma$. Similarly, $\mathcal{I} \models \phi \leq \gamma$ iff $\mathcal{I}(\phi) \leq \gamma$.

3. Fuzzy $GZ\ \mathcal{SROIQ}$

In this section, we define $GZ\ \mathcal{SROIQ}$, a fuzzy extension of $\mathcal{SROIQ}$ with the following features:

- Concepts denote fuzzy sets of individuals.
- Roles denote fuzzy binary relations.
- Axioms are extended to the fuzzy case, and some of them hold to a degree.
- The fuzzy connectives include Gödel and Zadeh fuzzy logics.
- There is a finite chain of degrees of truth.

We will assume the reader to be familiar with classical DLs.[4]

3.1. *Definition*

Notation. Firstly, let us introduce some notation that will be used in the paper. In the syntax of the logic, C, D are (possibly complex) concepts, A is an atomic concept, R is a (possibly complex) role, R_A is an atomic role, S is a simple role (see the definition below), and a, b are individuals.

We assume a finite chain of degrees of truth $\mathcal{N}$ and define $\mathcal{N}^+ = \mathcal{N} \setminus \{\gamma_0\}$. Degrees of truth will be denoted as $\gamma \in \mathcal{N}$ and $\alpha \in \mathcal{N}^+$. We will also define $+\gamma_i = \gamma_{i+1}$, $-\gamma_i = \gamma_{i-1}$.

In the fuzzy axioms, we will use $\bowtie \in \{\geq, >, \leq, <\}$, $\rhd \in \{\geq, >\}$, $\lhd \in \{\leq, <\}$. The symmetric $\bowtie^-$, and the negation $\neg \bowtie$ of an operator $\bowtie$ are defined as follows:

$\bowtie$	$\bowtie^-$	$\neg \bowtie$
$\geq$	$\leq$	$<$
$>$	$<$	$\leq$
$\leq$	$\geq$	$>$
$<$	$>$	$\geq$

Table 2. Syntax of fuzzy concepts and fuzzy roles in $GZ\ \mathcal{SROIQ}$.

Concept	**Syntax**
Top concept	$\top$
Bottom concept	$\bot$
Atomic concept	A
Conjunction	$C \sqcap D$
Disjunction	$C \sqcup D$
Gödel negation	$\neg_G C$
Zadeh negation	$\neg_Z C$
Gödel universal restriction	$\forall_G R.C$
Zadeh universal restriction	$\forall_Z R.C$
Existential restriction	$\exists R.C$
Fuzzy nominal	$\{\alpha/a\}$
At-least restriction	$\geq m\ S.C$
Gödel at-most restriction	$\leq_G n\ S.C$
Zadeh at-most restriction	$\leq_Z n\ S.C$
Local reflexivity	$\exists S.\mathtt{Self}$
Cut concept	$[C \geq \alpha]$
Role	**Syntax**
Atomic role	R_A
Universal role	U
Inverse role	R^-
Cut role	$[R \geq \alpha]$

We will use $\equiv$ to denote semantic equivalence. Finally, if an individual a is related with an individual b via a role R, we will say that b is an R-successor of a.

Syntax. Fuzzy $\mathcal{SROIQ}$ assumes three alphabets of symbols for concepts, roles and individuals. The syntax of fuzzy concepts and roles is shown in Table 2.

Example 1. Human and Young are atomic fuzzy concepts. isFriendOf is an atomic fuzzy role. $\mathsf{Human} \sqcap \mathsf{Young}$ denotes the fuzzy concept of young human. $[\mathsf{isFriendOf} \geq \mathtt{closeToTrue}]$ denotes the pairs of individuals which are close to be friends.

Remark 1. As opposed to the crisp case, there are two types of negations, universal restrictions and at-most restrictions, one corresponding to Gödel fuzzy logic and another one corresponding to Zadeh fuzzy logic (denoted with the subscripts $_G$ and $_Z$, respectively). However, there is only one type of conjunction, disjunction, existential or at-least restrictions because the semantics in both logics coincide.

Table 3. Syntax of axioms in $GZ\ \mathcal{SROIQ}$.

ABox axiom	**Syntax**
Concept assertion	$\langle a\!:\!C \bowtie \gamma\rangle$
Role assertion	$\langle (a,b)\!:\!R \bowtie \gamma\rangle$
Gödel negated role assertion	$\langle (a,b)\!:\!\neg_G R \bowtie \gamma\rangle$
Zadeh negated role assertion	$\langle (a,b)\!:\!\neg_Z R \bowtie \gamma\rangle$
Inequality assertion	$a \neq b$
Equality assertion	$a = b$
TBox axiom	**Syntax**
Gödel General Concept Inclusion (GCI)	$\langle C \sqsubseteq_G D \triangleright \gamma\rangle$
Zadeh General Concept Inclusion (GCI)	$\langle C \sqsubseteq_Z D \triangleright \gamma\rangle$
RBox axiom	**Syntax**
Gödel Role Inclusion Axiom (RIA)	$\langle R_1 \dots R_m \sqsubseteq_G R \triangleright \gamma\rangle$
Zadeh Role Inclusion Axiom (RIA)	$\langle R_1 \dots R_m \sqsubseteq_Z R \triangleright \gamma\rangle$
Transitive role axiom	$\mathtt{trans}(R)$
Disjoint role axiom	$\mathtt{dis}(S_1, S_2)$
Reflexive role axiom	$\mathtt{ref}(R)$
Irreflexive role axiom	$\mathtt{irr}(S)$
Symmetric role axiom	$\mathtt{sym}(R)$
Asymmetric role axiom	$\mathtt{asy}(S)$

Another difference with respect to the classical case is the presence of fuzzy nominals,[23] as well as cut concepts and roles.[38]

Remark 2. Fuzzy nominals of the form $\{\alpha_1/o_1,\ \alpha_2/o_2, \dots, \alpha_n/o_n\}$ can be represented as $\{\alpha_1/o_1\} \sqcup \{\alpha_2/o_2\} \sqcup \dots \sqcup \{\alpha_n/o_n\}$.

A *Fuzzy Knowledge Base* (KB) contains a finite set of axioms organized in three parts: a fuzzy ABox $\mathcal{A}$ (axioms about individuals), a fuzzy TBox $\mathcal{T}$ (axioms about concepts) and a fuzzy RBox $\mathcal{R}$ (axioms about roles).

The axioms in our logic are shown in Table 3. A *fuzzy axiom* is an axiom that has a truth degree in $\mathcal{N}$. We will assume that fuzzy KBs do not include fuzzy axioms of the forms $\langle\tau \geq \gamma_0\rangle, \langle\tau \leq \gamma_p\rangle, \langle\tau < \gamma_0\rangle, \langle\tau > \gamma_p\rangle$. The reason is that the two former axioms are tautologies and the two latter axioms are always contradictions.

Given a fuzzy KB $\mathcal{K}$, we say that a is a *new individual* and that C is a *new concept* if they do not appear in $\mathcal{K}$.

Example 2. The axiom $\langle \mathtt{paul}\!:\!\mathtt{Tall} \geq \mathtt{closeToTrue}\rangle$ states that it is likely true that Paul can be considered tall. $\langle \mathtt{isFriendOf\ isFriendOf} \sqsubseteq_G \mathtt{isFriendOf} \geq \mathtt{closeToFalse}\rangle$ states that the friends of my friends can also my considered as my friends with at least a low degree.

Remark 3. Note again the difference with respect to the classical case as there are two types of negated role assertions, namely fuzzy GCIs and fuzzy RIAs.

As in the crisp case, there are some restrictions in the use of roles in order to guarantee the decidability of the logic:

- Concepts $\geq m\ S.C, \leq_G n\ S.C, \leq_Z n\ S.C, \exists S.\texttt{Self}$ require simple roles.
- RBox axioms $\texttt{dis}(S_1, S_2), \texttt{irr}(S), \texttt{asy}(S)$ require simple roles.
- RBox axioms cannot contain the universal role U.
- Finally, given a regular order $\prec$[b], every RIA should be $\prec$-regular. A RIA $\langle w \sqsubseteq_I R \rhd \gamma \rangle$ is $\prec$-*regular*, where $I \in \{G, Z\}$, if R is atomic and:
  - — $w = RR$, or
  - — $w = R^-$, or
  - — $w = S_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or
  - — $w = RS_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or
  - — $w = S_1 \dots S_n R$ and $S_i \prec R$ for all $i = 1, \dots, n$.

Simple roles are inductively defined as follows:

- R_A is simple if it does not occur on the right side of a fuzzy RIA.
- R^- is simple if R is.
- R is simple if every fuzzy RIAs such that R occurs on its right side is of the form $\langle S \sqsubseteq_I R\ \rhd \gamma \rangle$, for a simple role S and $I \in \{G, Z\}$.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

- Every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.
- Every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow \mathcal{N}$.
- Every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{N}$.

We use $\otimes$ for denoting Gödel t-norm (minimum), $\oplus$ for Gödel t-conorm (maximum), $\ominus_G$ for Gödel negation, $\ominus_Z$ for the involutive negation, $\Rightarrow_G$ for Gödel implication, and $\Rightarrow_Z$ for Kleene-Dienes implication.

The fuzzy interpretation function is extended to fuzzy *complex concepts, roles* and *axioms* as shown in Tables 4 and 5, respectively.

$C^{\mathcal{I}}$ denotes the membership function of the fuzzy concept C with respect to the fuzzy interpretation $\mathcal{I}$. $C^{\mathcal{I}}(x)$ gives us the degree of being x an element of the fuzzy concept C under $\mathcal{I}$. Similarly, $R^{\mathcal{I}}$ denotes the membership function of the fuzzy role R with respect to $\mathcal{I}$. $R^{\mathcal{I}}(x, y)$ gives us the degree of being (x, y) an element of the fuzzy role R.

[b]A strict partial order $\prec$ on a set A is an irreflexive and transitive relation on A. A strict partial order $\prec$ on a set of roles $\mathbf{R}$ is called a *regular order* if it also satisfies $S \prec R \Leftrightarrow S^- \prec R, \forall R, S \in \mathbf{R}$.

Table 4. Semantics of fuzzy concepts and fuzzy roles in $GZ\ \mathcal{SROIQ}$.

Concept	**Semantics**
$(\top)^{\mathcal{I}}(x)$	γ_p
$(\bot)^{\mathcal{I}}(x)$	γ_0
$(A)^{\mathcal{I}}(x)$	$A^{\mathcal{I}}(x)$
$(C \sqcap D)^{\mathcal{I}}(x)$	$C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
$(C \sqcup D)^{\mathcal{I}}(x)$	$C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$
$(\neg_G C)^{\mathcal{I}}(x)$	$\ominus_G C^{\mathcal{I}}(x)$
$(\neg_Z C)^{\mathcal{I}}(x)$	$\ominus_Z C^{\mathcal{I}}(x)$
$(\forall_G R.C)^{\mathcal{I}}(x)$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow_G C^{\mathcal{I}}(y)\}$
$(\forall_Z R.C)^{\mathcal{I}}(x)$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow_Z C^{\mathcal{I}}(y)\}$
$(\exists R.C)^{\mathcal{I}}(x)$	$\sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
$(\{\alpha/a\})^{\mathcal{I}}(x)$	α if $x = a^{\mathcal{I}}$, γ_0 otherwise
$(\geq m\ S.C)^{\mathcal{I}}(x)$	$\sup_{y_1,\ldots,y_m \in \Delta^{\mathcal{I}}} \{$ $(\min_{i=1}^{m} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \bigotimes (\otimes_{j<k} \{y_j \neq y_k\})\}$
$(\leq_G n\ S.C)^{\mathcal{I}}(x)$	$\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} \{$ $(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow_G (\oplus_{j<k} \{y_j = y_k\})\}$
$(\leq_Z n\ S.C)^{\mathcal{I}}(x)$	$\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} \{$ $(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow_Z (\oplus_{j<k} \{y_j = y_k\})\}$
$(\exists S.\mathtt{Self})^{\mathcal{I}}(x)$	$S^{\mathcal{I}}(x,x)$
$([C \geq \alpha])^{\mathcal{I}}(x)$	γ_p if $C^{\mathcal{I}}(x) \geq \alpha$, γ_0 otherwise
Role R	**Semantics**
$(R_A)^{\mathcal{I}}(x,y)$	$R_A^{\mathcal{I}}(x,y)$
$(U)^{\mathcal{I}}(x,y)$	γ_p
$(R^-)^{\mathcal{I}}(x,y)$	$R^{\mathcal{I}}(y,x)$
$([R \geq \alpha])^{\mathcal{I}}(x,y)$	γ_p if $R^{\mathcal{I}}(x,y) \geq \alpha$, γ_0 otherwise

Remark 4. Note an important difference with respect to previous works in fuzzy DLs. Usually, $\cdot^{\mathcal{I}}$ maps every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$, and every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$. Consequently, a fuzzy KB $\{\langle a : C > 0.5\rangle, \langle a : C < 0.75\rangle\}$ is satisfiable, by taking $C^{\mathcal{I}}(a) \in (0.5, 0.75)$. But now, given $\mathcal{N} = \{\mathtt{false}, \mathtt{closeToFalse}, \mathtt{neutral}, \mathtt{closeToTrue}, \mathtt{true}\}$, a fuzzy KB $\{\langle a : C > \mathtt{closeToFalse}\rangle, \langle a : C < \mathtt{neutral}\rangle\}$ is not satisfiable, since $C^{\mathcal{I}}(a) \in \mathcal{N}$.

Table 5. Syntax and semantics of axioms in $GZ\ \mathcal{SROIQ}$.

ABox axiom	**Semantics**
$\langle a : C \bowtie \gamma \rangle$	$C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$
$\langle (a,b) : R \bowtie \gamma \rangle$	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \gamma$
$\langle (a,b) : \neg_G R \bowtie \gamma \rangle$	$\ominus_G R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$
$\langle (a,b) : \neg_Z R \bowtie \gamma \rangle$	$\ominus_Z R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$
$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$
$a = b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
TBox axiom	**Semantics**
$\langle C \sqsubseteq_G D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow_G D^{\mathcal{I}}(x)\} \rhd \gamma$
$\langle C \sqsubseteq_Z D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(x) \Rightarrow_Z D^{\mathcal{I}}(x)\} \rhd \gamma$
RBox axiom	**Semantics**
$\langle R_1 \dots R_m \sqsubseteq_G R \rhd \gamma \rangle$	$\inf_{x_1, x_{m+1} \in \Delta^{\mathcal{I}}} \{\sup_{x_2 \dots x_m \in \Delta^{\mathcal{I}}} \{ R_1^{\mathcal{I}}(x_1, x_2) \otimes \dots \otimes R_m^{\mathcal{I}}(x_m, x_{m+1})\} \Rightarrow_G R^{\mathcal{I}}(x_1, x_{n+1})\} \rhd \gamma$
$\langle R_1 \dots R_m \sqsubseteq_Z R \rhd \gamma \rangle$	$(\inf_{x_1, x_{m+1} \in \Delta^{\mathcal{I}}} \{\sup_{x_2 \dots x_m \in \Delta^{\mathcal{I}}} \{ R_1^{\mathcal{I}}(x_1, x_2) \otimes \dots \otimes R_m^{\mathcal{I}}(x_m, x_{m+1})\} \Rightarrow_Z R^{\mathcal{I}}(x_1, x_{m+1})\} \rhd \gamma$
$\texttt{trans}(R)$	$\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$
$\texttt{dis}(S_1, S_2)$	$\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = \gamma_0$ or $S_2^{\mathcal{I}}(x,y) = \gamma_0$
$\texttt{ref}(R)$	$\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = \gamma_p$
$\texttt{irr}(S)$	$\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = \gamma_0$
$\texttt{sym}(R)$	$\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$
$\texttt{asy}(S)$	$\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > \gamma_0$ then $S^{\mathcal{I}}(y,x) = \gamma_0$

Witnessed models. In order to correctly manage infima and suprema in fuzzy DLs, the notion of *witnessed* models appear.[39] A fuzzy interpretation $\mathcal{I}$ is *witnessed* iff the supremum of every formulae coincides with the minimum and the infimum of every formulae coincides with the maximum.

It is well known that every finite model is witnessed. Our logic enjoys WMP, because the number of degrees of truth in the models $\mathcal{N}$ is finite and the fuzzy operators are closed under $\mathcal{N}$.

Reasoning tasks. In the following, we will only consider fuzzy KB satisfiability, since (as in the crisp case) most inference problems can be reduced to it.[40]

- *Fuzzy KB satisfiability.* A fuzzy interpretation $\mathcal{I}$ *satisfies* (is a model of) a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

- *Entailment*: A fuzzy concept assertion $\langle a : C \geq \gamma \rangle$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \geq \gamma \rangle$) iff $\mathcal{K} \cup \{\langle a : C < \gamma \rangle\}$ is unsatisfiable. Similarly, $\mathcal{K} \models \langle (a, b) : R \geq \gamma \rangle$ iff $\mathcal{K} \cup \{\langle (a, b) : R < \gamma \rangle\}$ is unsatisfiable.
- *Fuzzy concept satisfiability.* C is γ-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle a : C \geq \gamma \rangle\}$ is satisfiable, where a is a new individual.
- *Fuzzy concept subsumption.* In Zadeh fuzzy logic, D subsumes C with degree γ $(C \sqsubseteq D \geq \gamma)$ w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{a : \neg C \sqcup D < \gamma\}$ is unsatisfiable, where a is a new individual.

  In Gödel fuzzy logic, D subsumes C with degree γ $(C \sqsubseteq_G D \geq \gamma)$ w.r.t. a fuzzy KB $\mathcal{K}$ iff, for every $\alpha \in \mathcal{N}^+$ with $\alpha \leq \gamma$, $\mathcal{K} \cup \{\langle a : C \geq \alpha \rangle\} \cup \{\langle a : D < \alpha \rangle\}$ is unsatisfiable, where a is a new individual.
- *Greatest lower bound.* The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle \tau \geq \alpha \rangle\}$. It can be computed by performing at most $\log |\mathcal{N}|$ entailment tests.[40]

It can be easily shown that fuzzy *GZ* $\mathcal{SROIQ}$ is a sound extension of crisp $\mathcal{SROIQ}$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{\gamma_0 = 0, \gamma_p = 1\}$.

Proposition 1. *GZ* $\mathcal{SROIQ}$ *fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to* $\{\gamma_0 = 0, \gamma_p = 1\}$.

3.2. *Logical properties*

The following properties are extensions to a finite chain $\mathcal{N}$ of properties for *G* $\mathcal{SROIQ}$ and *Z* $\mathcal{SROIQ}$.

(1) *Concept simplification*:

  (a) $C \sqcap \top \equiv C$,
  (b) $C \sqcup \bot \equiv C$,
  (c) $C \sqcap \bot \equiv \bot$,
  (d) $C \sqcup \top \equiv \top$,
  (e) $\exists R.\bot \equiv \bot$,
  (f) $\forall_G R.\top \equiv \top$,
  (g) $\forall_Z R.\top \equiv \top$.

(2) *Involutive* negation:

  (a) $\neg_Z \neg_Z C \equiv C$,
  (b) $\neg_G \neg_G C \not\equiv C$.

(3) *Excluded middle and contradiction*:

  (a) $C \sqcup \neg_Z C \not\equiv \top$,
  (b) $C \sqcap \neg_Z C \not\equiv \bot$,
  (c) $C \sqcup \neg_G C \not\equiv \bot$,
  (d) $C \sqcap \neg_G C \equiv \bot$.

(4) *Idempotence* of conjunction and disjunction:

  (a) $C \sqcap C \equiv C$,
  (b) $C \sqcup C \equiv C$.

(5) *De Morgan* laws:

  (a) $\neg_G(C \sqcup D) \equiv \neg_G C \sqcap \neg_G D$,
  (b) $\neg_G(C \sqcap D) \equiv \neg_G C \sqcup \neg_G D$,
  (c) $\neg_Z(C \sqcap D) \equiv \neg_Z C \sqcup \neg_Z D$,
  (d) $\neg_Z(C \sqcup D) \equiv \neg_Z C \sqcap \neg_Z D$.

(6) *Inter-definability of concepts*:

  (a) $\bot \equiv \neg_G \top$,
  (b) $\bot \equiv \neg_Z \top$,
  (c) $\top \equiv \neg_G \bot$,
  (d) $\top \equiv \neg_Z \bot$,
  (e) $C \sqcap D \equiv \neg_Z(\neg_Z C \sqcup \neg_Z D)$,
  (f) $C \sqcap D \not\equiv \neg_G(\neg_G C \sqcup \neg_G D)$,
  (g) $C \sqcup D \equiv \neg_Z(\neg_Z C \sqcap \neg_Z D)$,
  (h) $C \sqcup D \not\equiv \neg_G(\neg_G C \sqcap \neg_G D)$,
  (i) $\exists R.C \equiv \neg_Z \forall_Z R.(\neg_Z C)$,
  (j) $\exists R.C \not\equiv \neg_G \forall_G R.(\neg_G C)$,
  (k) $\exists R.(\neg_Z C) \equiv \neg_Z \forall_Z R.C$,
  (l) $\exists R.(\neg_G C) \not\equiv \neg_G \forall_G R.C$,
  (m) $\forall_Z R.C \equiv \neg_Z \exists R.(\neg_Z C)$,
  (n) $\forall_G R.C \not\equiv \neg_G \exists R.(\neg_G C)$,
  (o) $\forall_Z R.(\neg_Z C) \equiv \neg_Z \exists R.C$,
  (p) $\forall_G R.(\neg_G C) \equiv \neg_G \exists R.C$,
  (q) $\exists S.C \equiv \geq 1\ S.C$,
  (r) $\geq m\ S.C \equiv \neg_Z(\leq_Z m-1\ S.C)$.
  (s) $\geq m\ S.C \not\equiv \neg_G(\leq_G m-1\ S.C)$,
  (t) $\leq_Z n\ S.C \equiv \neg_Z(\geq n+1\ S.C)$,
  (u) $\leq_G n\ S.C \equiv \neg_G(\geq n+1\ S.C)$.

(7) *Inter-definability of axioms*:

  (a) $\langle \tau \geq \gamma \rangle \equiv \langle \tau > +\gamma \rangle$,
  (b) $\langle \tau < \alpha \rangle \equiv \langle \tau \leq -\alpha \rangle$,
  (c) $\langle a : C \geq \alpha \rangle \equiv \langle \{\alpha/a\} \sqsubseteq_G C \geq \gamma_p \rangle$,
  (d) $\langle (a,b) : \neg_Z R \bowtie \gamma \rangle \equiv \langle (a,b) : R \bowtie^- \ominus_Z \gamma \rangle$,
  (e) $\texttt{irr}(S) \equiv \langle \top \sqsubseteq_G \neg \exists S.\texttt{Self} \geq \gamma_p \rangle$,
  (f) $\texttt{trans}(R) \equiv \langle RR \sqsubseteq_G R \geq \gamma_p \rangle$,
  (g) $\texttt{sym}(R) \equiv \langle R^- \sqsubseteq_G R \geq \gamma_p \rangle$.

(8) *Modus tolens*:

  (a) $C \sqsubseteq_Z D \equiv \neg_Z D \sqsubseteq_Z \neg_Z C$.

(b) $C \sqsubseteq_G D \not\equiv \neg_G D \sqsubseteq_G \neg_G C$.

(9) *Modus ponens*:

(a) $\langle a : C \rhd \gamma_i \rangle$ and $\langle C \sqsubseteq_G D \rhd \gamma_j \rangle$ imply $\langle a : D \rhd \gamma_i \otimes \gamma_j \rangle$.
(b) $\langle a : C \rhd \gamma_i \rangle$ and $\langle C \sqsubseteq_Z D \rhd \gamma_j \rangle$ imply $\langle a : D \rhd \gamma_j \rangle$ if $\gamma_i + \rhd \ \ominus_Z \gamma_j$.
(c) $\langle (a, b) : R \rhd \gamma_i \rangle$ and $\langle R \sqsubseteq_G R' \rhd \gamma_j \rangle$ imply $\langle (a, b) : R' \rhd \gamma_i \otimes \gamma_j \rangle$.
(d) $\langle (a, b) : R \rhd \gamma_i \rangle$ and $\langle R \sqsubseteq_Z R' \rhd \gamma_j \rangle$ imply $\langle (a, b) : R' \rhd \gamma_j \rangle$ if $\gamma_i + \rhd \ \ominus_Z \gamma_j$.
(e) $\langle (a, b) : R \rhd \gamma_i \rangle$ and $\langle a : \forall_G R.C \rhd \gamma_j \rangle$ imply $\langle b : C \rhd \gamma_i \otimes \gamma_j \rangle$.
(f) $\langle (a, b) : R \rhd \gamma_i \rangle$ and $\langle a : \forall_Z R.C \rhd \gamma_j \rangle$ imply $\langle b : C \rhd \gamma_j \rangle$ if $\gamma_i + \rhd \ \ominus_Z \gamma_j$.

(10) Gödel GCIs and RIAs *chaining*:

(a) $\langle C \sqsubseteq_G D \rhd \gamma_1 \rangle$ and $\langle D \sqsubseteq_G E \rhd \gamma_2 \rangle$ imply $\langle C \sqsubseteq_G E \rhd \gamma_1 \otimes \gamma_2 \rangle$.
(b) $\langle R \sqsubseteq_G R' \rhd \gamma_1 \rangle$ and $\langle R' \sqsubseteq_G R'' \rhd \gamma_2 \rangle$ imply $\langle R \sqsubseteq_G R'' \rhd \gamma_1 \otimes \gamma_2 \rangle$.

(11) *Counter-intuitive effects* of Zadeh GCIs and RIAs:

(a) A concept does not fully subsume itself in general, i.e., $(C \sqsubseteq_Z C)^{\mathcal{I}} \neq \gamma_p$.
(b) Similarly, a role does not fully subsume itself in general, i.e., $(R \sqsubseteq_Z R)^{\mathcal{I}} \neq \gamma_p$.
(c) Crisp concept subsumption forces concepts to be crisp, i.e., $\langle C \sqsubseteq_Z D \geq \gamma_p \rangle \Rightarrow \inf_{x \in \Delta^{\mathcal{I}}} \max\{\ominus_Z C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \geq \gamma_p$ which is true iff for each element x of the domain $D^{\mathcal{I}}(x) = \gamma_p$ or $C^{\mathcal{I}}(x) = \gamma_0$.
(d) Similarly, crisp role subsumption forces roles to be crisp, i.e., $\langle R_1 \sqsubseteq_Z R_2 \geq \gamma_p \rangle$ is true iff for each pair of elements x, y of the domain $R_2^{\mathcal{I}}(x, y) = \gamma_p$ or $R_1^{\mathcal{I}}(x, y) = \gamma_0$.

Remark 5. Properties (7*a*) and (7*b*) show that it is enough to restrict to fuzzy axioms of the forms $\langle \tau \geq \gamma \rangle$ and $\langle \tau \leq \gamma \rangle$.

Table 6 compares $GZ\ \mathcal{SROIQ}$ with other fuzzy DLs by studying whether they satisfy the previous properties or not. In particular, we consider Zadeh (Z), Gödel (G), and Łukasiewicz (Ł) using the respective fuzzy operators in Tables 4 and 5. In Table 6, + means that the property is satisfied, − means that it is not, and ? means that it is only satisfied if a finite chain of degrees of truth $\mathcal{N}$ is considered. For Zadeh, Gödel and Łukasiewicz, we can either assume a finite chain of degrees or not — it is interesting to notice that only properties (7a) and (7b) depend on this assumption to hold.

4. A Crisp Representation for *GZ* $\mathcal{SROIQ}$

In this section we prove that reasoning with $GZ\ \mathcal{SROIQ}$ is decidable by showing how to reduce a fuzzy KB in this logic into an equivalent crisp KB, in such a way that existing $\mathcal{SROIQ}$ reasoners could be applied to the resulting KB. This process only makes sense if $p > 1$, because $p = 1$ makes the logic crisp.

The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some

Table 6. Properties of different fuzzy DLs.

Property	GZ	Z	G	Ł	Property	GZ	Z	G	Ł
(1a)	+	+	+	+	(6l)	+	−	+	−
(1b)	+	+	+	+	(6m)	+	+	−	+
(1c)	+	+	+	+	(6n)	+	−	+	−
(1d)	+	+	+	+	(6o)	+	+	+	+
(1e)	+	+	+	+	(6p)	+	+	+	+
(1f)	+	+	+	+	(6q)	+	+	+	+
(1g)	+	+	+	+	(6r)	+	+	−	+
(2a)	+	+	−	+	(6s)	+	−	+	−
(2b)	+	−	+	−	(6t)	+	+	+	+
(3a)	+	+	+	−	(6u)	+	+	+	+
(3b)	+	+	−	−	(7a)	+	?	?	?
(3c)	+	+	+	−	(7b)	+	?	?	?
(3d)	+	−	+	+	(7c)	+	−	+	+
(4a)	+	+	+	−	(7d)	+	+	−	+
(4b)	+	+	+	−	(7e)	+	−	+	+
(5a)	+	+	+	+	(7f)	+	−	+	+
(5b)	+	+	+	+	(7g)	+	−	+	+
(5c)	+	+	+	+	(8a)	+	+	−	+
(5d)	+	+	+	+	(8b)	+	−	+	−
(6a)	+	+	+	+	(9a)	+	−	+	+
(6b)	+	+	+	+	(9b)	+	+	−	−
(6c)	+	+	+	+	(9c)	+	−	+	+
(6d)	+	+	+	+	(9d)	+	+	−	−
(6e)	+	+	−	+	(9e)	+	−	+	+
(6f)	+	−	+	−	(9f)	+	+	−	−
(6g)	+	+	−	+	(10a)	+	−	+	−
(6h)	+	−	+	−	(10b)	+	−	+	−
(6i)	+	+	−	+	(11a)	+	+	−	−
(6j)	+	−	+	−	(11b)	+	+	−	−
(6k)	+	+	−	+	(11c)	+	+	−	−
(6l)	+	−	+	−	(11d)	+	+	−	−
(6m)	+	+	−	+					

new axioms are added to preserve the semantics of concepts and relations. Finally, every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using the new crisp elements.

4.1. *Adding new elements*

Let $\mathbf{A}$ be the set of atomic fuzzy concepts and $\mathbf{R}$ the set of atomic fuzzy roles in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$, respectively.

Now, for each $\alpha \in \mathcal{N}^+$, for each $A \in \mathbf{A}$, a new atomic concepts $A_{\geq\alpha}$ is introduced. $A_{\geq\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α, i.e, the α-cut of A. Similarly, for each $R_A \in \mathbf{R}$, a new atomic role $R_{A\geq\alpha}$ is created.

Remark 6. Note that the atomic elements $A_{\geq\gamma_0}$ and $R_{A\geq\gamma_0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB (e.g. we do not allow GCIs of the form $C \sqsubseteq_G D \geq \gamma_0$).

Remark 7. Note also that, as opposite to previous works,[23,35,36,41] we are not introducing elements of the forms $A_{>\gamma}$ and $R_{>\gamma}$ (for each $\gamma \in \mathcal{N} \setminus \{\gamma_p\}$), since now $A_{>\gamma_i}$ is equivalent to $A_{\geq\gamma_{i+1}}$, and $R_{>\gamma_i}$ is equivalent to $R_{\geq\gamma_{i+1}}$.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq p-1$ and for each $A \in \mathbf{A}$, $T(\mathcal{N})$ is the smallest terminology containing these axioms:

$$A_{\geq\gamma_{i+1}} \sqsubseteq A_{\geq\gamma_i} \tag{1}$$

Similarly, for each $R_A \in \mathbf{R}$, $R(\mathcal{N})$ is the smallest terminology containing:

$$R_{A\geq\gamma_{i+1}} \sqsubseteq R_{A\geq\gamma_i} \tag{2}$$

Remark 8. Again, note that the number of new axioms needed here is less than the number needed in similar works,[23,35,36,41] since we do not need to deal with elements of the forms $A_{>\gamma}$ and $R_{>\gamma}$.

Example 3. Let us assume $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$ and consider a fuzzy KB $\mathcal{K}$ that represents the following knowledge:

- John's friends may be experienced drivers:

$$\langle \mathsf{john}\colon \forall_G \mathsf{hasFriend}.\mathsf{ExperiencedDriver} \geq \texttt{neutral} \rangle$$

- Experienced drivers are likely to not be young people:

$$\langle \mathsf{ExperiencedDriver} \sqsubseteq_G (\neg_Z \mathsf{YoungPerson}) \geq \texttt{closeToTrue} \rangle$$

- One should not be considered a friend of himself:

$$\texttt{irr}(\mathsf{hasFriend})$$

In the first axiom, we use $\forall_G$ for illustrative purposes, although $\forall_Z$ could also be valid. Regarding the second axiom, on the one hand, neither ExperiencedDriver nor YoungPerson should be interpreted as crisp concepts, so we use $\sqsubseteq_G$ rather than $\sqsubseteq_Z$. On the other hand, non young people ($\neg$YoungPerson) should be interpreted as a crisp concept, so we use $\neg_Z$ rather than $\neg_G$. Furthermore, it is interesting to reflect that $\neg_Z\neg_Z\mathsf{YoungPerson} \equiv \mathsf{YoungPerson}$.

Now, we create some new elements and some axioms to presever their semantics. $T(\mathcal{N})$ contains the new axioms due to the new concepts:

$$\begin{aligned}
&\mathsf{YoungPerson}_{\geq \mathtt{true}} \sqsubseteq \mathsf{YoungPerson}_{\geq \mathtt{closeToTrue}}\\
&\mathsf{YoungPerson}_{\geq \mathtt{closeToTrue}} \sqsubseteq \mathsf{YoungPerson}_{\geq \mathtt{neutral}}\\
&\mathsf{YoungPerson}_{\geq \mathtt{neutral}} \sqsubseteq \mathsf{YoungPerson}_{\geq \mathtt{closeToFalse}}\\
&\mathsf{ExperiencedDriver}_{\geq \mathtt{true}} \sqsubseteq \mathsf{ExperiencedDriver}_{\geq \mathtt{closeToTrue}}\\
&\mathsf{ExperiencedDriver}_{\geq \mathtt{closeToTrue}} \sqsubseteq \mathsf{ExperiencedDriver}_{\geq \mathtt{neutral}}\\
&\mathsf{ExperiencedDriver}_{\geq \mathtt{neutral}} \sqsubseteq \mathsf{ExperiencedDriver}_{\geq \mathtt{closeToFalse}}
\end{aligned}$$

The case for the roles is similar, with $R(\mathcal{N})$ containing the following axioms:

$$\begin{aligned}
&\mathsf{hasFriend}_{\geq \mathtt{true}} \sqsubseteq \mathsf{hasFriend}_{\geq \mathtt{closeToTrue}}\\
&\mathsf{hasFriend}_{\geq \mathtt{closeToTrue}} \sqsubseteq \mathsf{hasFriend}_{\geq \mathtt{neutral}}\\
&\mathsf{hasFriend}_{\geq \mathtt{neutral}} \sqsubseteq \mathsf{hasFriend}_{\geq \mathtt{closeToFalse}}
\end{aligned}$$

4.2. *Mapping fuzzy concepts, roles and axioms*

Fuzzy concept and role expressions are reduced by using mapping ρ, as shown in Table 7. Given a fuzzy concept C, $\rho(C, \geq \alpha)$ is a crisp set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C, \bowtie \gamma)$ are similar. ρ is defined in a similar way for fuzzy roles and this equivalence also holds. It can be verified that $\rho(C, \bowtie \gamma) \equiv \neg\rho(C, \neg \bowtie \gamma)$.

Remark 9. Notice that the reduction of the fuzzy nominal is different to that presented in previous work.[23,35,36] Actually, the reduction presented there $(\rho(\{\alpha_1/o_1, \ldots, \alpha_m/\ o_m\}, \lhd\gamma) = \{o_i \mid \alpha_i \leq \gamma, 1 \leq i \leq n\})$ is defective. These papers consider complex fuzzy nominals of the form $\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}$ and degrees of truth in $[0, 1]$, but the same problem appears under our new framework (see Appendix for details and a formal proof of the correctness of our reduction).

Example 4. Consider an interpretation domain $\Delta^{\mathcal{I}} = \{o_1, o_2, o_3\}$ and a concept $C = \{0.5/o_1, 0.7/o_2\}$. According to previous works,[23,35,36] $\rho(C, \leq 0.6) = \{o_1\}$. However, this is not a correct reduction, because we cannot conclude that $\rho(C, \leq 0.6)$ is equivalent to a concept composed of only individual $\{o_1\}$, as nothing prevents that o_3 also belongs to this concept. A correct reduction would be $\neg\{o_2\}$.

Mapping ρ deserves some comments. Firstly, it is interesting to remark that $\rho(A, \leq \gamma)$ is different from $\rho(\neg_Z A, \geq \gamma)$ and from $\rho(\neg_G A, \geq \gamma)$. Secondly, due to the restrictions in the definition of the fuzzy KB, some expressions cannot appear during the process:

- For any concept C and any role R, $\rho(C, \geq \gamma_0), \rho(C, > \gamma_p), \rho(C, \leq \gamma_p), \rho(C, < \gamma_0)$, $\rho(R, \geq \gamma_0), \rho(R, > \gamma_p), \rho(R, \leq \gamma_p), \rho(R, < \gamma_0)$ cannot appear due to the existing restrictions on the degree of the axioms in the fuzzy KB.
- $\rho(U, \lhd\gamma)$ can only appear in a negated role assertion.

Table 7. Mapping of concept and role expressions.

$\rho(\top, \rhd\gamma)$	$= \top$
$\rho(\top, \lhd\gamma)$	$= \bot$
$\rho(\bot, \rhd\gamma)$	$= \bot$
$\rho(\bot, \lhd\gamma)$	$= \top$
$\rho(A, \geq \gamma)$	$= A_{\geq\gamma}$
$\rho(A, > \gamma)$	$= A_{\geq+\gamma}$
$\rho(A, \leq \gamma)$	$= \neg A_{\geq+\gamma}$
$\rho(A, < \gamma)$	$= \neg A_{\geq\gamma}$
$\rho(\neg_G C, \rhd\gamma)$	$= \rho(C, \leq \gamma_0)$
$\rho(\neg_G C, \lhd\gamma)$	$= \rho(C, > \gamma_0)$
$\rho(\neg_Z C, \bowtie \gamma)$	$= \rho(C, \bowtie^- \ominus_Z \gamma)$
$\rho(C \sqcap D, \rhd\gamma)$	$= \rho(C, \rhd\gamma) \sqcap \rho(D, \rhd\gamma)$
$\rho(C \sqcap D, \lhd\gamma)$	$= \rho(C, \lhd\gamma) \sqcup \rho(D, \lhd\gamma)$
$\rho(C \sqcup D, \rhd\gamma)$	$= \rho(C, \rhd\gamma) \sqcup \rho(D, \rhd\gamma)$
$\rho(C \sqcup D, \lhd\gamma)$	$= \rho(C, \lhd\gamma) \sqcap \rho(D, \lhd\gamma)$
$\rho(\forall_G R.C, \geq \gamma)$	$= \bigsqcap_{\alpha \in \mathcal{N}^+ \mid \alpha \leq \gamma} (\forall \rho(R, \geq \alpha).\rho(C, \geq \alpha))$
$\rho(\forall_G R.C, > \gamma)$	$= \bigsqcap_{\alpha \in \mathcal{N}^+ \mid \alpha \leq +\gamma} (\forall \rho(R, \geq \alpha).\rho(C, \geq \alpha))$
$\rho(\forall_G R.C, \leq \gamma)$	$= \bigsqcup_{\alpha \in \mathcal{N} \mid \alpha \leq \gamma} (\exists \rho(R, > \alpha).\rho(C, \leq \alpha))$
$\rho(\forall_G R.C, < \gamma)$	$= \bigsqcup_{\alpha \in \mathcal{N}^+ \mid \alpha \leq \gamma} (\exists \rho(R, \geq \alpha).\rho(C, < \alpha))$
$\rho(\forall_Z R.C, \geq \gamma)$	$= \forall \rho(R, > \ominus_Z \gamma).\rho(C, \geq \gamma)$
$\rho(\forall_Z R.C, > \gamma)$	$= \forall \rho(R, \geq \ominus_Z \gamma).\rho(C, > \gamma)$
$\rho(\forall_Z R.C, \lhd\gamma)$	$= \exists \rho(R, \lhd^- \ominus_Z \gamma).\rho(C, \lhd\gamma)$
$\rho(\exists R.C, \rhd\gamma)$	$= \exists \rho(R, \rhd\gamma).\rho(C, \rhd\gamma)$
$\rho(\exists R.C, \lhd\gamma)$	$= \forall \rho(R, \neg\lhd \gamma).\rho(C, \lhd\gamma)$
$\rho(\{\alpha/a\}, \rhd\gamma)$	$= \{a\}$ if $\alpha \rhd \gamma$, $\bot$ otherwise
$\rho(\{\alpha/a\}, \lhd\gamma)$	$= \neg\{a\}$ if $\alpha \neg \lhd \gamma$, $\top$ otherwise
$\rho(\geq m\ S.C, \rhd\gamma)$	$= \geq m\ \rho(S, \rhd\gamma).\rho(C, \rhd\gamma)$
$\rho(\geq m\ S.C, \lhd\gamma)$	$= \leq m{-}1\ \rho(S, \neg\lhd\ \gamma).\rho(C, \neg\lhd\ \gamma)$
$\rho(\leq_G n\ S.C, \rhd\gamma)$	$= \leq n\ \rho(S, > \gamma_0).\rho(C, S, > \gamma_0)$
$\rho(\leq_G n\ S.C, \lhd\gamma)$	$= \geq n+1\ \rho(S, > \gamma_0).\rho(C, S, > \gamma_0)$
$\rho(\leq_Z n\ S.C, \geq \gamma)$	$= \leq n\ \rho(S, > \ \ominus_Z \gamma).\rho(C, > \ \ominus_Z \gamma)$
$\rho(\leq_Z n\ S.C, > \gamma)$	$= \leq n\ \rho(S, \geq \ \ominus_Z \gamma).\rho(C, \geq \ \ominus_Z \gamma)$
$\rho(\leq_Z n\ S.C, \lhd\gamma)$	$= \geq n+1\ \rho(S, \lhd^- \ \ominus_Z \gamma).\rho(C, \lhd^- \ \ominus_Z \gamma)$
$\rho(\exists S.\mathtt{Self}, \rhd\gamma)$	$= \exists \rho(S, \rhd\gamma).\mathtt{Self}$
$\rho(\exists S.\mathtt{Self}, \lhd\gamma)$	$= \neg\exists \rho(S, \neg\lhd \gamma).\mathtt{Self}$
$\rho([C \geq \alpha], \rhd\gamma)$	$= \rho(C, \geq \alpha)$
$\rho([C \geq \alpha], \lhd\gamma)$	$= \rho(C, < \alpha)$
$\rho(R_A, \geq \gamma)$	$= R_{A\geq\gamma}$
$\rho(R_A, > \gamma)$	$= R_{A\geq+\gamma}$
$\rho(R_A, \leq \gamma)$	$= \neg R_{A\geq+\gamma}$
$\rho(R_A, < \gamma)$	$= \neg R_{A\geq\gamma}$
$\rho(U, \rhd\gamma)$	$= U$
$\rho(U, \lhd\gamma)$	$= \neg U$
$\rho(R^-, \bowtie \gamma)$	$= \rho(R, \bowtie \gamma)^-$
$\rho([R \geq \alpha], \rhd\gamma)$	$= \rho(R, \geq \alpha)$
$\rho([R \geq \alpha], \lhd\gamma)$	$= \rho(R, < \alpha)$
$\rho(\neg_G R, \rhd\gamma)$	$= \rho(R, \leq \gamma_0)$
$\rho(\neg_G R, \lhd\gamma)$	$= \rho(R, > \gamma_0)$
$\rho(\neg_Z R, \bowtie \gamma)$	$= \rho(R, \bowtie^- \ominus_Z \gamma)$

Table 8. Reduction of the axioms.

$$\begin{aligned}
\kappa(\langle a{:}C \bowtie \gamma\rangle) &= \{a{:}\rho(C, \bowtie \gamma)\}\\
\kappa(\langle (a,b){:}R \bowtie \gamma\rangle) &= \{(a,b){:}\rho(R, \bowtie \gamma)\}\\
\kappa(\langle (a,b){:}\neg_G R \bowtie \gamma\rangle) &= \{(a,b){:}\rho(\neg_G R, \bowtie \gamma)\}\\
\kappa(\langle (a,b){:}\neg_Z R \bowtie \gamma\rangle) &= \{(a,b){:}\rho(\neg_Z R, \bowtie \gamma)\}\\
\kappa(a \neq b) &= \{a \neq b\}\\
\kappa(a = b) &= \{a = b\}\\
\kappa(\langle C \sqsubseteq_G D \geq \gamma\rangle) &= \textstyle\bigcup_{\alpha \in \mathcal{N}^+ \mid \alpha \leq \gamma} \{\rho(C, \geq \alpha) \sqsubseteq \rho(D, \geq \alpha)\}\\
\kappa(\langle C \sqsubseteq_G D > \gamma\rangle) &= \textstyle\bigcup_{\alpha \in \mathcal{N}^+ \mid \alpha \leq^+ \gamma} \{\rho(C, \geq \alpha) \sqsubseteq \rho(D, \geq \alpha)\}\\
\kappa(\langle C \sqsubseteq_Z D \geq \gamma\rangle) &= \{\rho(C, > \ominus_Z \gamma) \sqsubseteq \rho(D, \geq \gamma)\}\\
\kappa(\langle C \sqsubseteq_Z D > \gamma\rangle) &= \{\rho(C, \geq \ominus_Z \gamma) \sqsubseteq \rho(D, > \gamma)\}\\
\kappa(\langle R_1 \ldots R_n \sqsubseteq_G R \geq \gamma\rangle) &= \textstyle\bigcup_{\alpha \in \mathcal{N}^+ \mid \alpha \leq \gamma} \{\rho(R_1, \geq \alpha) \ldots \rho(R_n, \geq \alpha) \sqsubseteq \rho(R, \geq \alpha)\}\\
\kappa(\langle R_1 \ldots R_n \sqsubseteq_G R > \gamma\rangle) &= \textstyle\bigcup_{\alpha \in \mathcal{N}^+ \mid \alpha \leq^+ \gamma} \{\rho(R_1, \geq \alpha) \ldots \rho(R_n, \geq \alpha) \sqsubseteq \rho(R, \geq \alpha)\}\\
\kappa(\langle R_1 \ldots R_m \sqsubseteq_Z R \geq \gamma\rangle) &= \{\rho(R_1, > \ominus_Z \gamma) \ldots \rho(R_m, > \ominus_Z \gamma) \sqsubseteq \rho(R, \geq \gamma)\}\\
\kappa(\langle R_1 \ldots R_m \sqsubseteq_Z R > \gamma\rangle) &= \{\rho(R_1, \geq \ominus_Z \gamma) \ldots \rho(R_m, \geq \ominus_Z \gamma) \sqsubseteq \rho(R, > \gamma)\}\\
\kappa(\mathtt{trans}(R)) &= \textstyle\bigcup_{\gamma \in \mathcal{N}^+} \{\mathtt{trans}(\rho(R, \geq \gamma))\}\\
\kappa(\mathtt{dis}(S_1, S_2)) &= \{\mathtt{dis}(\rho(S_1, > \gamma_0), \rho(S_2, > \gamma_0))\}\\
\kappa(\mathtt{ref}(R)) &= \{\mathtt{ref}(\rho(R, \geq \gamma_p))\}\\
\kappa(\mathtt{irr}(S)) &= \{\mathtt{irr}(\rho(S, > \gamma_0))\}\\
\kappa(\mathtt{sym}(R)) &= \textstyle\bigcup_{\gamma \in \mathcal{N}^+} \{\mathtt{sym}(\rho(R, \geq \gamma))\}\\
\kappa(\mathtt{asy}(S)) &= \{\mathtt{asy}(\rho(S, > \gamma_0))\}
\end{aligned}$$

Axioms are reduced as in Table 8, where $\kappa(\tau)$ maps a fuzzy axiom τ in $GZ\ \mathcal{SROIQ}$ into a set of crisp axioms in $\mathcal{SROIQ}$.

We note $\kappa(\mathcal{A})$ the union of the reductions of all the fuzzy axioms in $\mathcal{A}$. Analogously, $\kappa(\mathcal{T})$ is the union of the reductions of all fuzzy concepts in $\mathcal{T}$, where as $\kappa(\mathcal{R})$ is the union of the reductions of all fuzzy roles in $\mathcal{R}$.

Let $\mathsf{crisp}(\mathcal{K})$ denote the reduction of a fuzzy ontology $\mathcal{K}$. A fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$ is reduced into a KB $\mathsf{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{N}) \cup \kappa(\mathcal{T}), R(\mathcal{N}) \cup \kappa(\mathcal{R})\rangle$.

Example 5. Consider again Example 3 and let us map the three axioms in $\mathcal{K}$.

- $\kappa(\langle \mathsf{john}{:}\forall_G \mathsf{hasFriend}.\mathsf{ExperiencedDriver} \geq \mathtt{neutral}\rangle) =$
  $$\begin{aligned}\mathsf{john}:\ & (\forall \rho(\mathsf{hasFriend}, \geq \mathtt{closeToFalse}).\rho(\mathsf{ExperiencedDriver}, \geq \mathtt{closeToFalse}))\\ & \sqcap(\forall \rho(\mathsf{hasFriend}, \geq \mathtt{neutral}).\rho(\mathsf{ExperiencedDriver}, \geq \mathtt{neutral}))\\ \mathsf{john}:\ & (\forall\, \mathsf{hasFriend}_{\geq \mathtt{closeToFalse}}.\mathsf{ExperiencedDriver}_{\geq \mathtt{closeToFalse}}) \sqcap\\ & (\forall\, \mathsf{hasFriend}_{\geq \mathtt{neutral}}.\mathsf{ExperiencedDriver}_{\geq \mathtt{neutral}})\end{aligned}$$
- $\kappa(\langle \mathsf{ExperiencedDriver} \sqsubseteq_G (\neg_Z \mathsf{YoungPerson}) \geq \mathtt{closeToTrue}\rangle) =$

$$\begin{gathered}\{\rho(\mathsf{ExperiencedDriver}, \geq \mathtt{closeToFalse}) \sqsubseteq \rho(\neg_Z \mathsf{YoungPerson}, \geq \mathtt{closeToFalse})\}\\ \cup \{\rho(\mathsf{ExperiencedDriver}, \geq \mathtt{neutral}) \sqsubseteq \rho(\neg_Z \mathsf{YoungPerson}, \geq \mathtt{neutral})\} \cup\\ \{\rho(\mathsf{ExperiencedDriver}, \geq \mathtt{closeToTrue}) \sqsubseteq \rho(\neg_Z \mathsf{YoungPerson}, \geq \mathtt{closeToTrue})\} =\end{gathered}$$

$\{\mathsf{ExperiencedDriver}_{\geq \texttt{closeToFalse}} \sqsubseteq \rho(\mathsf{YoungPerson}_{\leq \texttt{closeToTrue}})\} \cup$
$\{\mathsf{ExperiencedDriver}_{\geq \texttt{neutral}} \sqsubseteq \rho(\mathsf{YoungPerson}_{\leq \texttt{neutral}})\} \cup$
$\{\mathsf{ExperiencedDriver}_{\geq \texttt{closeToTrue}} \sqsubseteq \rho(\mathsf{YoungPerson}_{\leq \texttt{closeToFalse}})\} =$
$\{\mathsf{ExperiencedDriver}_{\geq \texttt{closeToFalse}}.\neg\mathsf{YoungPerson}_{\geq \texttt{true}})\} \cup$
$\{\mathsf{ExperiencedDriver}_{\geq \texttt{neutral}}.\neg\mathsf{YoungPerson}_{\geq \texttt{closeToTrue}})\} \cup$
$\{\mathsf{ExperiencedDriver}_{\geq \texttt{closeToTrue}}.\neg\mathsf{YoungPerson}_{\geq \texttt{neutral}})\}$

- $\kappa(\texttt{irr}(\mathsf{hasFriend})) = \texttt{irr}(\mathsf{hasFriend}_{\geq \texttt{closeToFalse}})$

4.3. *Allowing crisp concepts and roles*

It is easy to see that the complexity of the crisp representation is caused by fuzzy atomic concepts and roles. Fortunately, in real applications not all concepts and roles will be fuzzy. Therefore, an interesting optimization is enabling to specify that an atomic concept or an atomic role is crisp.

Let us suppose that A is a fuzzy atomic concept. Then, we need p concepts of the form $A_{\geq \alpha}$ to represent it, as well as $p-1$ axioms to preserve their semantics. On the other hand, if A is declared to be crisp, we just need one crisp concept A_{crisp} to represent it and no new axioms. The case for atomic roles R_A is similar, thus needing only one crisp element R_{crisp} in the reduction.

Handling these crisp elements is very easy, since we only need to extend ρ by considering those elements asserted to be interpreted as crisp as shown in Table 9.

Table 9. Reduction of crisp concepts and roles.

$\rho(A, \rhd\gamma) = A_{\mathsf{crisp}}$
$\rho(A, \lhd\gamma) = \neg A_{\mathsf{crisp}}$
$\rho(R_A, \rhd\gamma) = R_{\mathsf{crisp}}$
$\rho(R_A, \lhd\gamma) = \neg R_{\mathsf{crisp}}$

4.4. *Properties of the reduction*

Correctness. Firstly, we highlight that the reduction preserves simplicity of the roles and regularity of the RIAs. That said, the following theorem shows that the logic is decidable and that the reductions preserves reasoning.

Theorem 1. *A GZ $\mathcal{SROIQ}$ fuzzy KB $\mathcal{K}$ is satisfiable iff its crisp representation $\mathsf{crisp}(\mathcal{K})$ is satisfiable.*

The proof can easily be obtained by merging similar proofs for *G* $\mathcal{SROIQ}$[35] and *Z* $\mathcal{SROIQ}$,[23] with the only change that the finite set of degrees of truth in $[0,1]$ is now a finite chain $\mathcal{N}$ such that the minimum and the maximum elements of the chain are equivalent to 0 (false) and 1 (true). In the Appendix we will discuss and formally prove a more general result, stated in the next subsection.

Corollary 1. *The following problems are decidable in GZ $\mathcal{SROIQ}$: fuzzy KB satisfiability, entailment, fuzzy concept satisfiability, fuzzy concept subsumption, and greatest lower bound.*

Complexity. The *depth of a fuzzy concept* is inductively defined as follows:

- $\mathsf{depth}(A) = \mathsf{depth}(\top) = \mathsf{depth}(\bot) = \mathsf{depth}(\{\alpha/a\}) = \mathsf{depth}(\exists S.\mathtt{Self}) = 1$,
- $\mathsf{depth}(\neg_G C) = \mathsf{depth}(\neg_Z C) = \mathsf{depth}(\exists R.C) = \mathsf{depth}(\forall_G R.C) = \mathsf{depth}(\forall_Z R.C) = \mathsf{depth}(\geq m\ S.C) = \mathsf{depth}(\leq_G n\ S.C) = \mathsf{depth}(\leq_Z n\ S.C) = \mathsf{depth}(C \geq \alpha]) = 1 + \mathsf{depth}(C)$,
- $\mathsf{depth}(C \sqcap D) = \mathsf{depth}(C \sqcup D) = 1 + max\{\mathsf{depth}(C), \mathsf{depth}(D)\}$,

The depth of a crisp concept is defined analogously. Recalling that $p + 1$ is the number of degrees of truth in the chain, it is easy to see that:

- Every fuzzy concept expression of depth k generates a crisp concept expression of depth k except Gödel universal restrictions.
- The reduction of a Gödel universal restriction $\forall_G R.C$ of depth k generates a crisp concept expression of depth $k+1$. In the worst case, the size is $\mathcal{O}(|C||\mathcal{N}|^k)$. For instance, for $C = \forall R.(\forall P.(\forall Q.A))$, $k = 3$.
- Most axioms of the fuzzy KB generate one crisp axiom. However, some of them (Gödel GCIs, Gödel RIAs, transitive role axioms and symmetric role axioms) generate several crisp axioms.
  - — $|\kappa(\mathcal{A})| = |\mathcal{A}|$,
  - — $|\kappa(\mathcal{T})| \leq p \cdot |\mathcal{T}|$,
  - — $|\kappa(\mathcal{R})| \leq p \cdot |\mathcal{R}|$.
- Let $\mathbf{F_c}$ and $\mathbf{F_r}$ be the set of fuzzy concepts and fuzzy roles in $\mathcal{K}$, respectively. In order to preserve the semantics of the new elements, we are also introducing some new crisp axioms.
  - — $|T(\mathcal{N})| = (p-1) \cdot |\mathbf{F_c}|$,
  - — $|R(\mathcal{N})| = (p-1) \cdot |\mathbf{F_r}|$.

Remark 10. Using a finite chain of degrees of truth with cardinality $p+1$ produces a smaller number of new crisp axioms. Let us recall what happens in the opposite case.[23,35,36] In this case, the crisp representation algorithm has to consider a set of relevant degrees of truth $\mathcal{N}' = \{0, 0.5, 1\} \cup \{\gamma | \langle \tau \bowtie \gamma \rangle \in \mathcal{K}\} \cup \{\gamma | \langle \tau \bowtie 1 - \gamma \rangle \in \mathcal{K}\}$. Nevertheless, if $p' = |\mathcal{N}'| - 1$, then the size is bounded by:

- $|T(\mathcal{N}')| = (2p' - 1) \cdot |\mathbf{F_c}|$,
- $|R(\mathcal{N}')| = (2p' - 1) \cdot |\mathbf{F_r}|$.

All in all, the size of the resulting KB ($|\mathsf{crisp}(\mathcal{K})|$) is $\mathcal{O}(|\mathcal{K}||\mathcal{N}|^k)$, where k is the maximal depth of the universal restriction concepts appearing in $\mathcal{K}$.

The responsible of such a high complexity is the constructor $\forall_G R.C$, which does not make it possible to infer the exact degrees of truth and requires building

Table 10. Reduction of some approximations of Gödel universal restrictions.

$\rho(\forall_G[R \geq \alpha_1].[C \geq \alpha_2], \rhd\gamma)$	$= \forall\rho(R, \geq \alpha_1).\rho(C, \geq \alpha_2)$
$\rho(\forall_G[R \geq \alpha_1].[C \geq \alpha_2], \lhd\gamma)$	$= \exists\rho(R, \geq \alpha_1).\rho(C, < \alpha_2)$
$\rho(\forall_G R.C), \rhd\gamma)$, with R crisp	$= \forall R_{\mathsf{crisp}}.\rho(C, \rhd\gamma)$
$\rho(\forall_G R.C), \lhd\gamma)$, with R crisp	$= \exists R_{\mathsf{crisp}}.\rho(C, \lhd\gamma)$
$\rho(\forall_G R.C), \rhd\gamma)$, with C crisp	$= \forall\rho(R, > 0).C_{\mathsf{crisp}}$
$\rho(\forall_G R.C), \lhd\gamma)$, with C crisp	$= \exists\rho(R, > 0).(\neg C_{\mathsf{crisp}})$

disjunctions or conjunctions over all possible combinations of the degrees of truth. However, in most of the cases Gödel universal restrictions can be approximated:

- A first possibility is to use cut concepts and roles, replacing $(\forall_G R.C)$ with $(\forall_G[R \geq \alpha_1].[C \geq \alpha_2])$, meaning that every individual which is related through role R with degree (at least) α_1 must belong to C with (at least) degree α_2.
- Another possibility is to assume that R is crisp.
- A final possibility is to assume that C is crisp. This is case is less interesting because $\forall_G R.C$ becomes a crisp concept as well.

The reduction of these approximations is detailed in Table 10. Whenever these approximations are possible, the resulting KB is $\mathcal{O}(|\mathcal{N}|)$.

Modularity. An interesting property is that the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included. From an implementation point of view, this property allows computing the reduction of the ontology off-line and updating $\mathsf{crisp}(\mathcal{K})$ incrementally.

Theorem 2. *Let $\mathcal{K}$ be a GZ $\mathcal{SROIQ}$ fuzzy knowledge base involving a set of fuzzy atomic roles $\mathbf{A}$ and a set of a set of atomic roles $\mathbf{R}$; let $\mathcal{N}$ be a finite chain of degrees of truth, and let τ be a GZ $\mathcal{SROIQ}$ axiom such that:*

(1) for every atomic concept A which appears in τ, $A \in \mathbf{A}$,
(2) for every atomic role R_A which appears in τ, $R_A \in \mathbf{R}$,
(3) if γ appears in τ, then $\gamma \in \mathcal{N}$.

Then, $\mathsf{crisp}(\mathcal{K} \cup \tau) = \mathsf{crisp}(\mathcal{K}) \cup \kappa(\tau)$.

The proof is trivial from the following observations:

- Every axiom is reduced to a combination of new crisp elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- τ does not introduce new atomic concepts, atomic roles nor new degrees.
- Every axiom is mapped independently from the others.

Theorem 2 assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often. These are reasonable assumptions because ontologies do not usually change once that their development has finished. Also, we have seen the set of degrees of truth to be considered for any reasoning task is $\mathcal{N}$.

4.5. *The case of other finite operators*

In the previous sections, we have defined a fuzzy DL joining the fuzzy operators in Gödel and Zadeh fuzzy logics, showing that the reduction of some operator into classical $\mathcal{SROIQ}$ does not interfere with the other operators. Hence, it is possible to extend the logic to use more operators.

In this section we will show that the reasoning algorithm can be adapted to any finite fuzzy logic with the involutive negation $\neg_Z$. Hence, our fuzzy DL can include any (possibly more than one) finite negation, t-norm, t-conorm and implication. We use the subscript $_F$ to indicate that the operator belongs to a fuzzy logic F and assume that the t-conorm is dual to the t-norm.

The algorithm to compute an equivalent crisp representation of a fuzzy ontology $\mathcal{K}$ is the same as the algorithm explained in Sec. 4, but changing the definition of the mappings ρ and κ to reflect the semantics of the new fuzzy operators. Firstly, let us introduce some necessary terms.

Definition 1. Let $\Rightarrow_F$ be a fuzzy implication defined in $\mathcal{N}$, $\gamma_x, \gamma_y \in \mathcal{N}$. Let $X \subseteq \mathcal{N} \times \mathcal{N}$ be a set of pairs of degrees of truth. We define the mappings $R(X)$ and $L(X)$ as follows:

- $R(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists(\gamma_x, \gamma'_y) \in X \text{ such that } \gamma'_y < \gamma_y\}$,
- $L(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists(\gamma'_x, \gamma_y) \in X \text{ such that } \gamma'_x < \gamma_x\}$.

The set $O_{\Rightarrow_F \geq \gamma}$ is defined as: $L(R(\{(\gamma_x, \gamma_y) \mid \gamma_x \Rightarrow_F \gamma_y \geq \gamma\}))$.

Example 6. Let $\mathcal{N} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ and let $\Rightarrow_G$ be finite Gödel implication. Then, $O_{\Rightarrow_G \geq \gamma_3} = \{(\gamma_0, \gamma_0), (\gamma_1, \gamma_1), (\gamma_2, \gamma_2), (\gamma_3, \gamma_3)\}$.

Note that $O_{\Rightarrow_F \geq \gamma}$ is a subset of the pairs of elements whose implication is at least γ and always contains the pair (γ_0, γ_0).

Definition 2. $O^+_{\Rightarrow_F \geq \gamma}$ denotes $O_{\Rightarrow_F \geq \gamma} \setminus \{(\gamma_0, \gamma_0)\}$.

Definition 3. Let $\Rightarrow_F$ be a fuzzy implication defined in $\mathcal{N}$, $\gamma_x, \gamma_y \in \mathcal{N}$. Let $X \subseteq \mathcal{N} \times \mathcal{N}$ be a set of pairs of degrees of truth. We define the mappings $L(X)$ as in Definition 1 and $R'(X)$ as follows:

- $R'(X) = \{(\gamma_x, \gamma_y) \in X \mid \nexists(\gamma_x, \gamma'_y) \in X \text{ such that } \gamma'_y > \gamma_y\}$.

The set $O_{\Rightarrow_F \leq \gamma}$ is defined as: $L(R'(\{(\gamma_x, \gamma_y) \mid \gamma_x \Rightarrow_F \gamma_y \leq \gamma\}))$.

Table 11. Differences in the reduction for other finite fuzzy operators.

$\rho(\neg_F C, \geq \gamma)$	$= \rho(C, \leq \gamma_x)$, where $\gamma_x = \max\{\alpha \in \mathcal{N} \mid \ominus_F \alpha \geq \gamma\}$
$\rho(\neg_F C, \leq \gamma)$	$= \rho(C, \geq \gamma_x)$, where $\gamma_x = \min\{\alpha \in \mathcal{N}^+ \mid \ominus_F \alpha \geq \gamma\}$
$\rho(C \sqcap_F D, \geq \gamma)$	$= \bigsqcup_{F\ \gamma_x, \gamma_y \in \mathcal{N} : \gamma_x \otimes_F \gamma_y \geq \gamma} \rho(C, \geq \gamma_x) \sqcap_F \rho(D, \geq \gamma_y)$
$\rho(C \sqcap_F D, \leq \gamma)$	$= \rho(\neg_Z(\neg_Z C \sqcup_F \neg_Z D), \leq \gamma)$
$\rho(C \sqcup_F D, \geq \gamma)$	$= \rho(C, \geq \gamma) \sqcup_F \rho(D, \geq \gamma)$
	$\bigsqcup_{F\ \gamma_x, \gamma_y \in \mathcal{N} : \gamma_x \oplus_F \gamma_y \geq \gamma} \rho(C, \geq \gamma_x) \sqcup_F \rho(D, \geq \gamma_y)$
$\rho(C \sqcup_F D, \leq \gamma)$	$= \rho(\neg_Z(\neg_Z C \sqcap_F \neg_Z D), \leq \gamma)$
$\rho(\forall_F R.C, \geq \gamma)$	$= \bigsqcup_{F\ (\gamma_x, \gamma_y) \in O_{\Rightarrow_F \geq \gamma}} \forall_F \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)$
$\rho(\forall_F R.C, \leq \gamma)$	$= \bigsqcup_{F\ (\gamma_x, \gamma_y) \in O_{\Rightarrow_F \leq \gamma}} \exists_F \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)$
$\rho(\exists_F R.C, \geq \gamma)$	$= \bigsqcup_{F\ \gamma_x, \gamma_y \in \mathcal{N} : \gamma_x \otimes_F \gamma_y \geq \gamma} \exists_F \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)$
$\rho(\exists_F R.C, \leq \gamma)$	$= \rho(\neg_Z \forall_{SF} R.(\neg_Z C), \leq \gamma)$
$\rho(\geq_F m\ S.C, \bowtie \gamma)$	$= \rho(\exists_F S.(C \sqcap_F B_1) \sqcap_F \ldots \sqcap_F \exists_F S.(C \sqcap_F B_m), \bowtie \gamma)$, where B_1, $\ldots, B_m$ are new fuzzy atomic concepts forming a partition
$\rho(\leq_F n\ S.C, \bowtie \gamma)$	$= \rho(\neg_{RF}(\geq_F n+1\ S.C), \bowtie \gamma)$
$\rho(\neg_F R, \geq \gamma)$	$= \rho(R, \leq \gamma_x)$, where $\gamma_x = \max\{\alpha \in \mathcal{N} \mid \ominus_F \alpha \geq \gamma\}$
$\rho(\neg_F R, \leq \gamma)$	$= \rho(R, \geq \gamma_x)$, where $\gamma_x = \min\{\alpha \in \mathcal{N}^+ \mid \ominus_F \alpha \geq \gamma\}$
$\kappa(\langle (a,b) : \neg_F R \bowtie \gamma \rangle)$	$= (a,b) : \rho(\neg_F R, \bowtie \gamma)$
$\kappa(\langle C \sqsubseteq_F D \geq \gamma \rangle)$	$= \rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y), \forall (\gamma_x, \gamma_y) \in O^+_{\Rightarrow_F \geq \gamma}$
$\kappa(\langle R_1 \ldots R_m \sqsubseteq_F R \geq \gamma \rangle)$	$= \rho(R_1, \geq \gamma_{x1}) \ldots \rho(R_m, \geq \gamma_{xm}) \sqsubseteq \rho(R, \geq \gamma_y)$, $\forall (\gamma_x, \gamma_y) \in O^+_{\Rightarrow_F \geq \gamma}$ and $\forall \gamma_{x1}, \ldots, \gamma_{xm} \in \mathcal{N}^+$ such that $\gamma_{x1} \otimes_F \ldots \otimes_F \gamma_{xm} \geq \gamma_x$

Definition 4. $\mathrm{T}(B_1, \ldots, B_m)$ is the set containing the following axioms:

(T_1) $\langle \top \sqsubseteq_F [B_i \geq \gamma_p] \sqcup_F [B_i \leq \gamma_0] \geq \gamma_p \rangle$, for $i \in \{1, \ldots, m\}$,
(T_2) $\langle \top \sqsubseteq_F B_1 \sqcup_F B_2 \sqcup_F \ldots \sqcup_F B_m \geq \gamma_p \rangle$,
(T_3) $\langle B_j \sqcap_F B_k \sqsubseteq_F \bot \geq \gamma_p \rangle\}$, for $i \in \{1, \ldots, m\}$ and $1 \leq j < k \leq m$.

$B_1, \ldots, B_m$ form a *partition* iff every fuzzy interpretation $\mathcal{I} \models \mathrm{T}(B_1, \ldots, B_m)$.

For simplicity and without any loss of generalization, we can use Properties (7a) and (7b) to ensure that the fuzzy ontology does not contain fuzzy axioms of the forms $\tau > \gamma$ or $\tau < \gamma$. Consequently, expressions of the form $\rho(C, > \gamma)$ and $\rho(C, < \gamma)$ do not need to be taken into account.

That said, Table 11 defines the differences in mappings ρ and κ with respect to Tables 7 and 8. $\Rightarrow_{SF}$ denotes an S-implication defined as $\gamma_x \Rightarrow_{SF} \gamma_y = \neg_Z(\gamma_x \otimes (\neg_Z \gamma_y))$ for a finite t-norm $\otimes$, and $\neg_{RF}$ denotes a residuated negation defined as $\neg_{RF}\gamma = \gamma \Rightarrow \gamma_0$ for some implication $\Rightarrow$.

It is not difficult to see that if we only consider the operators in Zadeh and Gödel fuzzy logics, then we get equivalent expressions as those detailed in Sec. 4. However, the expressions in Sec. 4 have been optimized by applying the following well-known equivalences over classical DLs:

- if $B_1 \sqsubseteq B_2$ then $B_1 \sqcup B_2 \sqcup \ldots B_m \equiv B_2 \sqcup \ldots B_m$,
- if $B_1 \sqsubseteq B_2$ then $B_1 \sqcap B_2 \sqcap \ldots B_m \equiv B_1 \sqcap \ldots B_m$.

Finally, the following results shows the correctness of our approach.

Theorem 3. *A finite fuzzy $\mathcal{SROIQ}$ fuzzy KB $\mathcal{K}$ is satisfiable iff its crisp representation $\mathsf{crisp}(\mathcal{K})$ is satisfiable.*

Proof. See Appendix. □

In this more general scenario, the size of the resulting classical ontology is higher than in the case of $GZ\ \mathcal{SROIQ}$ for several reasons. For instance, in this new case several concept expressions (conjunction, disjunction, cardinality restrictions ...) generate expressions of a higher depth than in the original fuzzy ontology.

5. Related Work

Since the first work of J. Yen in 1991,[42] an important number of fuzzy extensions to DLs can be found in the literature.[9] We will focus here on three common topics in the area: the study of different fuzzy logics within fuzzy DLs, the representation of fuzzy DLs using crisp DLs, and the use of a finite chain of degrees of truth.

Fuzzy logics in fuzzy DLs. While most of the research works in the are are limited to Zadeh fuzzy logic, there are several exceptions. Łukasiewicz fuzzy DLs have been studied[36,43–45] and implemented in the fuzzy DL reasoners FUZZYDL,[34] GERDS,[45] and YADLR.[46] A Gödel fuzzy DL has also been presented.[35] Previously, Gödel implication was used, but only in the semantics of GCIs and RIAs.[23]

There are also some attempts to reason with arbitrary continuous t-norms. P. Hájek studied fuzzy $\mathcal{ALC}$ under any continuous t-norm and reported a reasoning algorithm based on a reduction to fuzzy propositional logic.[39] Fuzzy $\mathcal{ALC}$ under arbitrary continuous t-norms extended with Łukasiewicz negation have been studied.[47] Both of these works are restricted to the witnessed models of fuzzy $\mathcal{ALC}$ without fuzzy GCIs. U. Straccia proposed a more expressive logic, fuzzy $\mathcal{SHOIN}(\mathbf{D})$, based on any t-norm, but without giving any reasoning algorithm.[48]

Nevertheless, the combination of different logics has not received such attention, and the present paper is the first theoretical work in this direction.

Crisp representations for fuzzy DLs. The first effort in this direction was a reasoning preserving procedure for fuzzy $Z\ \mathcal{ALC}$ into crisp $\mathcal{ALCH}$, under Zadeh logic.[33,41] A series of works widened the former work to $Z\ \mathcal{SROIQ}(\mathbf{D})$.[23,49]

A different approach considered a family of fuzzy DLs using α-cuts as atomic concept and roles.[50] The approach is slightly different to ours because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce a fuzzy $\mathcal{ALCQ}$ ABox[51] and a fuzzy $\mathcal{ALCH}$ concept[52] to their crisp versions. Both of these works assume an empty TBox.

There is also a proposal to represent every fuzzy set using two crisp sets (its support and its core) in $Z\ \mathcal{ALCIN}(\circ)$.[53] The authors commented the possibility of using more crisp sets, in order to have a more refined representation. Anyhow, there is a loss of information that does not occur in our approach.

Most of the previous work consider Zadeh fuzzy DLs, but Gödel ($G\ \mathcal{SROIQ}$)[35] and n-valued Łukasiewicz ($Ł_n\mathcal{SROIQ}$)[36] have also been studied.

It is also worth to stress out the crisp representation of two components of fuzzy DLs which are independent of the particular logic, namely fuzzy concrete domains[23] and modified fuzzy concepts and roles.[35]

Finite chains of degrees of truth in fuzzy DLs. U. Straccia proposed the use of truth values taken from an uncertainty lattice as degrees of truth, thus supporting quantitative and qualitative reasoning in fuzzy $Z\ \mathcal{ALC}$.[33] To guarantee soundness and completeness of the reasoning, the set of labels is assumed to be finite. A recent extension of this work by other authors considers $Z\ \mathcal{SHIN}$.[54] Nowadays, finite chains are receiving more attention to close the gap between mathematical fuzzy logic and fuzzy DLs.[26,55] Finally, there is a recent previous effort to provide a crisp representation for fuzzy $\mathcal{ALCH}$ based on a finite chain.[56]

6. Conclusions and Future Work

This paper has discussed fuzzy DLs with finite fuzzy operators corresponding to different fuzzy logics. These logics can be used as the theoretical basis of an extension of the language OWL 2 managing imprecise and vague knowledge.

Firstly, we have presented $GZ\ \mathcal{SROIQ}$, a fuzzy extension of the DL $\mathcal{SROIQ}$ joining the fuzzy operators from Gödel and Zadeh fuzzy logics. As opposed to the crisp case, there are two types of negations, universal restrictions and at-most restrictions, one corresponding to Gödel fuzzy logic and another one corresponding to Zadeh fuzzy logic. Our detailed study of the logical properties of the logic will help the ontology developers to use the connectives that better suit their needs.

The decidability of the logic has been shown by presenting a reasoning preserving reduction to the crisp case. We have also shown that a similar reduction to crisp $\mathcal{SROIQ}$ is possible when other finite fuzzy connectives are considered, but the resulting KB has a higher size than in the case of $GZ\ \mathcal{SROIQ}$.

We assume a finite chain of degrees of truth $\mathcal{N}$ such that the minimum and the maximum elements of the chain are equivalent to 0 (false) and 1 (true). This is very useful in practice since expert knowledge is usually expressed using linguistic terms, and since numerical interpretations of these labels can be avoided. In addition, it makes it possible to reuse the reduction of an ontology when adding new axioms, because in such case it is only necessary to include the reduction of the new axioms.

Due to the restriction of fuzzy interpretations to $\mathcal{N}$, there are some important differences in the reasoning procedure with respect to previous similar reductions. This reduction is implemented in the newest version of the fuzzy ontology reasoner DeLorean. The size of the resulting crisp KB is notably smaller when using a finite chain of degrees of truth. We have also presented some approximations of Gödel universal restrictions to obtain a smaller KB.

As future work, we would like to study an even more general framework to deal with finite fuzzy DLs, where the semantics any constructor would be defined by means of any family of functions over a finite totally ordered set of labels.

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Appendix A. Proof of Theorem 3

In this appendix we will proof one of the main results of this paper, Theorem 3. Before doing that, we will introduce some auxiliary results. Firstly, we will show that in the finite case, the satisfaction of an axiom can be reduced to computing which degrees make that possible.

Proposition 2. *Let* $\alpha, \beta \in \mathcal{N}, \gamma \in \mathcal{N}^+$. $(\alpha \otimes \beta) \geq \gamma \Leftrightarrow \exists \gamma_x, \gamma_y \in \mathcal{N}^+$ *such that* $\alpha \geq \gamma_x$, $\beta \geq \gamma_y$, *and* $\gamma_x \otimes \gamma_y \geq \gamma$.

Proof. The if direction is trivial by taking $\gamma_x = \alpha, \gamma_y = \beta$ and noting that $\alpha = \gamma_0$ or $\beta = \gamma_0$ are not possible (because they make $(\alpha \otimes \beta) = \gamma_0$ which contradicts $(\alpha \otimes \beta) \geq \gamma$), so $\gamma_x, \gamma_y \in \mathcal{N}^+$. The only if direction is easy to prove by using monotonicity of the t-norm. □

Proposition 3. *Let* $\alpha, \beta \in \mathcal{N}, \gamma \in \mathcal{N}^+$. $(\alpha \oplus \beta) \geq \gamma \Leftrightarrow \exists \gamma_x, \gamma_y \in \mathcal{N}$ *such that* $\alpha \geq \gamma_x$, $\beta \geq \gamma_y$, *and* $\gamma_x \oplus \gamma_y \geq \gamma$.

Proof. As in Proposition 2, but now $\alpha = \gamma_0$ or $\beta = \gamma_0$ are possible too. □

Proposition 4. *Let* $\alpha, \beta \in \mathcal{N}, \gamma \in \mathcal{N}^+$. $\alpha \Rightarrow \beta \geq \gamma \Leftrightarrow \forall (\gamma_x, \gamma_y) \in O^+_{\Rightarrow \geq \gamma}$, $\alpha \geq \gamma_x$ *implies* $\beta \geq \gamma_y$.

Proof. ($\Rightarrow$). Let $(\gamma_x, \gamma_y) \in O_{\Rightarrow \gamma}$ and assume that $\alpha \geq \gamma_x$. We will use reductio ad absurdum and show that $\beta < \gamma_y$ yields a contradiction. By Definition 1, $\beta < \gamma_y$ implies that $\gamma_x \Rightarrow \beta < \gamma$, because otherwise, $(\gamma_x, \beta) \in O^+_{\Rightarrow \gamma}$ and $(\gamma_x, \gamma_y) \notin O^+_{\Rightarrow \gamma}$. Hence, using that implications are non-decreasing in the second argument, we have $\alpha \Rightarrow \beta \leq \gamma_x \Rightarrow \beta < \gamma$, which is a contradiction.

(⇐). Let $\gamma_x = \alpha$. There are three different cases:

- $\gamma_x = \gamma_0$. Then, $\alpha \Rightarrow \beta = \gamma_0 \Rightarrow \beta = \gamma_p \geq \gamma$.
- $\gamma_x \neq \gamma_0$ and $(\gamma_x, \gamma_y) \in O^+_{\Rightarrow\geq\gamma}$ for some (and unique) $\gamma_y \in \mathcal{N}$. Then, $\gamma_x \Rightarrow \gamma_y \geq \gamma$ (as every pair of elements of $O^+_{\Rightarrow\geq\gamma}$) and $\alpha \geq \gamma_x$ implies $\beta \geq \gamma_y$ (by assumption). Now, $\alpha \Rightarrow \beta = \gamma_x \Rightarrow \beta \geq$(using that implications are non-decreasing in the second argument) $\gamma_x \Rightarrow \gamma_y \geq \gamma$.
- $\gamma_x \neq \gamma_0$ and $(\gamma_x, \gamma_y) \notin O^+_{\Rightarrow\geq\gamma}$ for any $\gamma_y \in \mathcal{N}$. By Definition 1, this can happen because *(i)* $\forall \gamma_y \in \mathcal{N}, \gamma_x \Rightarrow \gamma_y < \gamma$ or *(ii)* $\exists(\gamma'_x, \gamma_y) \in O^+_{\Rightarrow\geq\gamma}$ such that $\gamma'_x < \gamma_x$. *(i)* is not possible because $\gamma_x \Rightarrow \gamma_p = \gamma_p \geq \gamma$, so *(ii)* must be the reason. This implies that $(\gamma'_x \Rightarrow \gamma_y) \geq \gamma$. We have thus that $\alpha \geq \gamma_x > \gamma_x$ and, by assumption, $\beta \geq \gamma_y$. Now, $\alpha \Rightarrow \beta = \gamma_x \Rightarrow \beta \geq$ (using that implications are non-decreasing in the second argument) $\gamma_x \Rightarrow \gamma_y \geq \gamma$. □

Proposition 5. *Let $\alpha, \beta \in \mathcal{N}, \gamma \in \mathcal{N}^+$. $\alpha \Rightarrow \beta \leq \gamma \Leftrightarrow \exists(\gamma_x, \gamma_y) \in O_{\Rightarrow\leq\gamma}$ such that $\alpha \leq \gamma_x$ and $\beta \geq \gamma_y$.*

Proof. The if direction is trivial by taking $\gamma_x = \alpha, \gamma_y = \beta$ and noting that $\alpha = \gamma_0$ or $\beta = \gamma_0$ are not possible (because they make $(\alpha \otimes \beta) = \gamma_p$ which contradicts $(\alpha \Rightarrow \beta) \leq \gamma)$ because $\gamma < \gamma_p$, so $\gamma_x, \gamma_y \in \mathcal{N}^+$. The only if direction is easy using that fuzzy implications are non-increasing in the first argument and non-decreasing in the second one. □

Proposition 6. *Let $i \in \{1, \ldots, m\}, \alpha_i, \beta \in \mathcal{N}$ and $\gamma \in \mathcal{N}^+$. $(\otimes_i \alpha_i) \Rightarrow \beta \rhd \gamma \Leftrightarrow \forall(\gamma_x, \gamma_y) \in O^+_{\Rightarrow\geq\gamma}, \forall \gamma_{x_i} \in \mathcal{N}^+$ such that $\gamma_{x_1} \otimes \ldots \otimes \gamma_{x_m} \geq \gamma_x$, $\alpha_i \geq \gamma_{x_i}$ imply $\beta \geq \gamma_y$.*

Proof. Combine the proofs of Propositions 2 and 4. □

The next step is proving two equivalences that will be used in our reasoning algorithm. The first one considers at-least cardinality restrictions and existential restrictions.

Proposition 7. *Let $B_1, \ldots, B_m$ be fuzzy atomic concepts forming a partition and let $b_1, \ldots, b_m$ be individuals. $b_1, \ldots, b_m$ are pairwise different iff $B_i^{\mathcal{I}}(b_i) = \gamma_p$ holds for every $\mathcal{I}, i \in \{1, \ldots, m\}$.*

Proof. The if direction can be shown as follows. (T_1) imposes that $B_i^{\mathcal{I}} \in \{\gamma_0, \gamma_p\}$. Thus, (T_2) and (T_3) imply that every element of the domain verify $B_i^{\mathcal{I}}(b_i) = \gamma_p$ for some $i \in \{1, \ldots, m\}$ and $B_j^{\mathcal{I}}(b_i) = \gamma_0$ for every $j \in \{1, \ldots, m\}, j \neq i$. Hence, b_i must be pairwise different. The other direction can easily be proven. □

Proposition 8. *Let $B_1, \ldots, B_m$ be fuzzy atomic concepts forming a partition and let F be some finite fuzzy logic. The following equivalence holds:*

$$\geq_F m\, S.C \equiv \exists_F S.(C \sqcap_F B_1) \sqcap_F \ldots \sqcap_F \exists_F S.(C \sqcap_F B_m)$$

Proof. It follows easily from the semantics of the fuzzy concepts and the fact that, according to Proposition 7, all the S-successors are pairwise different. □

Now, let us show an equivalence between at-most and and-least cardinality restrictions:

Proposition 9. *Let F be some finite fuzzy logic and let $\neg_{RF}$ be the residuated negation of F, defined as $\neg_{RF}\gamma = \gamma \Rightarrow_F \gamma_0$, for every $\gamma \in \mathcal{N}$. The following equivalence holds:*
$\leq_F n\ S.C \equiv \neg_{RF}(\geq_F n+1\ S.C)$.

Proof. Let us analyze $(\leq_F\ n\ S.C)^{\mathcal{I}}(x)$. Firstly, note that $\oplus_{j<k}\{y_j = y_k\}$ can be either γ_0 or γ_p. On the one hand, if $y_1, \dots, y_{n+1}$ are not mutually different, then $\oplus_{j<k}\{y_j = y_k\} = \gamma_p$ and hence $\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\} \Rightarrow_F \gamma_p = \gamma_p$ for every implication F. On the other hand, if $y_1, \dots, y_{n+1}$ are mutually different, $\oplus_{j<k}\{y_j = y_k\} = \gamma_0$ and hence $\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\} \Rightarrow_F \gamma_0 = \ominus_{RF}(\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\})$. Hence, if the model does not have $n+1$ mutually disjoint individuals, then $(\leq_F\ n\ S.C)^{\mathcal{I}}(x) = \gamma_p$ and otherwise $(\leq_F n\ S.C)^{\mathcal{I}}(x) = \inf_{y_1,\dots,y_{n+1} \in \Delta^{\mathcal{I}}}\{\ominus_{RF}(\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\})\}$.

Now let us show that we obtain the same for $(\neg_{RF}(\geq_F\ n+1\ S.C))^{\mathcal{I}}(x) = \ominus_{RF}(\geq_F\ n+1\ S.C)^{\mathcal{I}}(x)$. Now, $\otimes_{F\ j<k}\{y_j \neq y_k\}$ can be either γ_0 or γ_p. On the one hand, if $y_1, \dots, y_{n+1}$ are not mutually different, $\otimes_{F\ j<k}\{y_j \neq y_k\} = \gamma_0$, so $\min_{i=1}^{m}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\} \bigotimes_F \gamma_0 = \gamma_0$ for every t-norm F. On the other hand, if $y_1, \dots, y_{n+1}$ are mutually different, we have $\min_{i=1}^{m}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\} \bigotimes_F \gamma_p = \min_{i=1}^{m}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\}$. Hence, if the model does not have $n+1$ mutually disjoint individuals, then $(\leq_F n\ S.C)^{\mathcal{I}}(x) = \gamma_0$ and otherwise $(\leq_F n\ S.C)^{\mathcal{I}}(x) = \sup_{y_1,\dots,y_{n+1} \in \Delta^{\mathcal{I}}}\{\min_{i=1}^{m}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\}\}$. Consequently, if the model does not have $n+1$ mutually disjoint individual $(\neg_{RF}(\leq_F n\ S.C))^{\mathcal{I}}(x) = \gamma_p$ and otherwise $(\neg_{RF}(\leq_F\ n\ \ S.C))^{\mathcal{I}}(x) = \inf_{y_1,\dots,y_{n+1} \in \Delta^{\mathcal{I}}}\{\ominus_{RF} \min_{i=1}^{m}\{S^{\mathcal{I}}(x, y_i) \otimes_F C^{\mathcal{I}}(y_i)\}\}$, which is the same that we obtained for $(\leq_F n\ S.C)^{\mathcal{I}}(x)$. □

At this point we are able to show the correctness of mapping ρ.

Definition 5. Given a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ we can define a (crisp) interpretation $\mathcal{I}_{\mathcal{C}} = \{\Delta^{\mathcal{I}c}, \cdot^{\mathcal{I}c}\}$ as follows:

- $\Delta^{\mathcal{I}c} = \Delta^{\mathcal{I}}$.
- $x^{\mathcal{I}c} = x^{\mathcal{I}}$, for all $x \in \Delta^{\mathcal{I}}$.
- $A_{\geq\alpha}^{\mathcal{I}c} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq \alpha\}$, for each $A \in \mathbf{A}, \alpha \in \mathcal{N}^+$.
- $R_{A\geq\alpha}^{\mathcal{I}c} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) \geq \alpha\}$, for each $R_A \in \mathbf{R}$, $\alpha \in \mathcal{N}^+$.

Proposition 10. *Let R be a fuzzy role, a, b individuals and $\mathcal{I}$ a fuzzy interpretation. Then:* $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \gamma \Leftrightarrow (a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}c}$.

Proof. By induction on the structure of fuzzy roles, the base case (atomic fuzzy roles) is trivial by construction of $\mathcal{I}_\mathcal{C}$ and in the other cases we have:

- *Inverse role.* Assume that $(R^-)^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \bowtie \gamma$. Then, $R^\mathcal{I}(b^\mathcal{I}, a^\mathcal{I}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}c}, a^{\mathcal{I}c}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}c}$. Consequently, $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in (\rho(R, \bowtie \gamma)^{\mathcal{I}c})^- \Leftrightarrow (b^{\mathcal{I}c}, a^{\mathcal{I}c}) \in \rho(R^-, \bowtie \gamma)^{\mathcal{I}c}$.
- *Universal role.* Assume that $U^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \gamma$. The universal verifies $U^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) = \gamma_p$. By definition of $\mathcal{I}_\mathcal{C}$, it follows that $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \Delta^{\mathcal{I}c} \times \Delta^{\mathcal{I}c}$ and consequently $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in U^{\mathcal{I}c}$. The case $U^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \leq \gamma$ is similar.
- *Cut role.* Assume that $([R \geq \alpha])^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \gamma$. Then, it follows that $([R \geq \alpha])^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) = \gamma_p$, which is the case if $R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \alpha$. By induction hypothesis, $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho(R, \geq \alpha)^{\mathcal{I}c} \Leftrightarrow (a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho([R \geq \alpha], \geq \gamma)^{\mathcal{I}c}$.

  Now assume that $([R \geq \alpha])^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \leq \gamma$. Then, it follows that $([R \geq \alpha])^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) = \gamma_0$, which is the case if $R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) < \alpha$. By induction hypothesis, $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho(R, < \alpha)^{\mathcal{I}c} \Leftrightarrow (a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho([R \geq \alpha], \lhd \gamma)^{\mathcal{I}c}$.
- *Role negation.* Assume that $\mathcal{I} \models \langle (a, b) : \neg_F R \geq \gamma \rangle$. Then, $\ominus_F R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \gamma$. Let $\gamma_x = \max\{\alpha \in \mathcal{N} \mid \ominus_F \alpha \geq \gamma\}$. Then, $R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \leq \gamma_x$ because $R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) > \gamma_x$ would imply $\ominus_F R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) < \gamma$ by definition of γ_x, which is a contradiction. By induction hypothesis, $(a^{\mathcal{I}c}, b^{\mathcal{I}c}) \in \rho(R, \leq \gamma_x)^{\mathcal{I}c}$ for some $\gamma_x \in \mathcal{N}^+$ such that $\ominus_F \gamma_x \geq \gamma$. This is true iff $\mathcal{I}_\mathcal{C} \models (a, b) : \rho(R, \leq \gamma_x) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle (a, b) : \neg_F R \geq \gamma \rangle)$. The case $\mathcal{I} \models \langle (a, b) : \neg_F R \leq \gamma \rangle$ is similar. □

Proposition 11. *Let C be a fuzzy concept, a and individual and $\mathcal{I}$ a fuzzy interpretation. Then: $C^\mathcal{I}(a^\mathcal{I}) \bowtie \gamma \Leftrightarrow a^I C \in \rho(C, \bowtie \gamma)^{\mathcal{I}c}$.*

Proof. By induction on the structure of fuzzy concepts, the base case (atomic fuzzy concepts) is trivial by construction of $\mathcal{I}_\mathcal{C}$ and in the other cases we have:

- *Top concept.* Assume that $\top^\mathcal{I}(a^\mathcal{I}) \geq \gamma$. By definition of $\mathcal{I}_\mathcal{C}$, it follows that $a^{\mathcal{I}c} \in \Delta^{\mathcal{I}c} = \top$. Consequently, $a^{\mathcal{I}c} \in \rho(\top, \geq \gamma)^{\mathcal{I}c}$. The case $\top^\mathcal{I}(a^\mathcal{I}) \leq \gamma$ is not possible because we do not allow axioms of the form $\langle \tau \leq \gamma_p \rangle$.
- *Bottom concept.* Use the equivalence $\bot \equiv \neg_F \top$.
- *Concept negation.* The case is similar to that of fuzzy role negation.
- *Conjunction.* Assume that $(C \sqcap_F D)^\mathcal{I}(a^\mathcal{I}) \geq \gamma$. Then, $(C^\mathcal{I}(a^\mathcal{I}) \otimes_F D^\mathcal{I}(a^\mathcal{I})) \geq \gamma$. By Proposition 2, $C^\mathcal{I}(a^\mathcal{I}) \geq \gamma_1$ and $D^\mathcal{I}(a^\mathcal{I}) \geq \gamma_2$ for some pair $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 \otimes_F \gamma_2 \geq \gamma$. By induction hypothesis, $a^{\mathcal{I}c} \in \rho(C, \geq \gamma_1)^{\mathcal{I}c}$ and $a^{\mathcal{I}c} \in \rho(D, \geq \gamma_2)^{\mathcal{I}c}$ for some pair $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 \otimes_F \gamma_2 \geq \gamma$. Thus, $a^{\mathcal{I}c} \in (\sqcup_{\gamma_1, \gamma_2 \in \mathcal{N}^+, \gamma_1 \otimes_F \gamma_2 \geq \gamma} (\rho(C, \geq \gamma_1) \sqcap \rho(D, \geq \gamma_2)))^{\mathcal{I}c} \Leftrightarrow a^{\mathcal{I}c} \in \rho(C \sqcap D, \geq \gamma)^{\mathcal{I}c}$.

  The case $(C \sqcap_F D)^\mathcal{I}(a^\mathcal{I})) \leq \gamma$ can be shown using the equivalence $C \sqcap_F D \equiv \neg_Z(\neg_Z C \sqcup_F \neg_Z D)$.
- *Disjunction.* Similar to the case of conjunction, but using Proposition 3.
- *Existential restriction.* Assume that $(\exists_F R.C)^\mathcal{I}(a^\mathcal{I}) \geq \gamma$. Then, $\sup_{b \in \Delta^\mathcal{I}}(R^\mathcal{I}(a^\mathcal{I}, b) \otimes_F C^\mathcal{I}(b)) \geq \gamma$. Our logic verifies the WMP. Hence, if this is true for the supremum, then there exists an individual b satisfying $(R^\mathcal{I}(a^\mathcal{I}, b) \otimes_F C^\mathcal{I}(b)) \geq \gamma$.

By Proposition 2, $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq \gamma_1$ and $C^{\mathcal{I}}(b) \geq \gamma_2$ for some pair $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 \otimes_F \gamma_2 \geq \gamma$. By induction hypothesis, $(a^{\mathcal{I}_\mathcal{C}}, b) \in \rho(R, \geq \gamma_1)^{\mathcal{I}_\mathcal{C}}$ and $b \in \rho(C, \geq \gamma_2)^{\mathcal{I}_\mathcal{C}}$ for some individual $b \in \Delta^{\mathcal{I}_\mathcal{C}}$ and some pair $\gamma_1, \gamma_2 \in \mathcal{N}^+$ such that $\gamma_1 \otimes_F \gamma_2 \geq \gamma$, which is equivalent to say that $a^{\mathcal{I}_\mathcal{C}} \in (\sqcup_{\gamma_1, \gamma_2 \in \mathcal{N}^+, \gamma_1 \otimes_F \gamma_2 \geq \gamma} (\exists \rho(R, \geq \gamma_1).\rho(C, \geq \gamma_2)))^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\exists_F R.C, \geq \gamma)^{\mathcal{I}_\mathcal{C}}$.

The case $(\exists_F R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \leq \gamma$ can be shown using $\exists_F R.C \equiv \neg_Z \forall_{SF} R.\neg_Z C$.

- *Universal restriction.* Assume that $(\forall_F R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma$. Then, $\inf_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow_F C^{\mathcal{I}}(b)\} \geq \gamma$. Since this is true for the infimum, an arbitrary individual $b \in \Delta^{\mathcal{I}}$ must satisfy that $(R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow_F C^{\mathcal{I}}(b)) \geq \gamma$. By Proposition 4, for every pair $(\gamma_x, \gamma_y) \in O^+_{\Rightarrow_F \geq \gamma}$, $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq \gamma_x$ implies that $C^{\mathcal{I}}(b) \geq \gamma_y$. By induction hypothesis, this implies that for every pair $(\gamma_x, \gamma_y) \in O^+_{\Rightarrow_F \geq \gamma}$ and every object $b \in \Delta^{\mathcal{I}_\mathcal{C}}$, $(a^{\mathcal{I}_\mathcal{C}}, b) \in \rho(R, \geq \gamma_x)^{\mathcal{I}_\mathcal{C}}$ implies that $b \in \rho(C, \geq \gamma_y)^{\mathcal{I}_\mathcal{C}}$. Consequently, $a^{\mathcal{I}_\mathcal{C}} \in (\sqcap_{\gamma_x, \gamma_y \in \mathcal{N}^+, (\gamma_x, \gamma_y) \in O^+_{\Rightarrow \geq \gamma}} (\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)))^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\forall_F R.C, \geq \gamma)^{\mathcal{I}_\mathcal{C}}$.

  Now assume that $(\forall_F R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \leq \gamma$. It follows that $\inf_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow_F C^{\mathcal{I}}(b)\} \geq \gamma$. Since the logic verifies the WMP, there is some $b \in \Delta^{\mathcal{I}}$ such that that $(R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow_F C^{\mathcal{I}}(b)) \leq \gamma$. By Proposition 5, $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leq \gamma_1$ and $C^{\mathcal{I}}(b^{\mathcal{I}}) \geq \gamma_2$ for some pair $\gamma_1, \gamma_2 \in O_{\Rightarrow_F \leq \gamma}$. By induction hypothesis, there is some $b \in \Delta^{\mathcal{I}_\mathcal{C}}$ such that $(a^{\mathcal{I}_\mathcal{C}}, b) \in \rho(R, \leq \gamma_1)^{\mathcal{I}_\mathcal{C}}$ and $b \in \rho(C, \geq \gamma_2)^{\mathcal{I}_\mathcal{C}}$ for some pair $\gamma_1, \gamma_2 \in O_{\Rightarrow_F \leq \gamma}$. Consequently, $a^{\mathcal{I}_\mathcal{C}} \in (\sqcup_{(\gamma_1, \gamma_2) \in O_{\Rightarrow_F \leq \gamma}} (\exists \rho(R, \leq \gamma_1).\rho(C, \geq \gamma_2)))^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\forall R.C, \leq \gamma)^{\mathcal{I}_\mathcal{C}}$.
- *Fuzzy nominal.* Assume that $(\{\alpha/o\})^{\mathcal{I}}(a) \geq \gamma$. Firstly, consider the case $\alpha \geq \gamma$, then it follows that $a^{\mathcal{I}} \in \{o\}^{\mathcal{I}}$ and that $(\{\alpha/o\})^{\mathcal{I}}(a)\} = \alpha \geq \gamma$. By construction of $\mathcal{I}_\mathcal{C}$, it holds that $a^{\mathcal{I}_\mathcal{C}} \in \{o\}^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\{\alpha/o\}, \geq \gamma)^{\mathcal{I}_\mathcal{C}}$. Now, let us consider the case $\alpha < \gamma$. Then $(\{\alpha/o\})^{\mathcal{I}}(a)\} = \gamma_0 = \bot^{\mathcal{I}}$. By construction of $\mathcal{I}_\mathcal{C}$, $\bot^{\mathcal{I}_\mathcal{C}}$ holds. Again, $a^{\mathcal{I}_\mathcal{C}} \in \rho(\{\alpha/o\}, \geq \gamma)^{\mathcal{I}_\mathcal{C}}$.

  The case $(\{\alpha/o\})^{\mathcal{I}}(a) \leq \gamma\rangle$ is different. Firstly, we consider the case $\alpha \leq \gamma$. Then, there are two possibilities: *(i)* $a^{\mathcal{I}} \in \{o\}^{\mathcal{I}}$ and $(\{\alpha/o\})^{\mathcal{I}}(a)\} = \alpha \leq \gamma$, or *(ii)* $a^{\mathcal{I}} \notin \{o\}^{\mathcal{I}}$ and $(\{\alpha/o\})^{\mathcal{I}}(a)\} = 0 \leq \gamma$. So, whatever $o^{\mathcal{I}}$ is, the axiom is satisfied. Clearly, $a^{\mathcal{I}} \in \top^{\mathcal{I}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \top^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\{\alpha/o\}, \leq \gamma)^{\mathcal{I}_\mathcal{C}}$. Now, consider the case $\alpha > \gamma$. It is necessary that $a^{\mathcal{I}} \notin \{o\}^{\mathcal{I}}$. By construction of $\mathcal{I}_\mathcal{C}$, it holds that $a^{\mathcal{I}_\mathcal{C}} \notin \{o\}^{\mathcal{I}_\mathcal{C}} \Leftrightarrow a^{\mathcal{I}_\mathcal{C}} \in \rho(\{\alpha/o\}, \leq \gamma)^{\mathcal{I}_\mathcal{C}}$.
- *At-least qualified number restriction.* It can be reduced to the case of existential restriction by using the equivalence in Proposition 8.
- *At-most qualified number restriction.* It can be reduced to the case of at-least qualified number restriction by using the equivalence in Proposition 9.
- *Local reflexivity.* Assume that $(\exists S.Self)^{\mathcal{I}}(a^I) \geq \gamma$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \geq \gamma$. By induction hypothesis, $(a^{\mathcal{I}_\mathcal{C}}, a^{\mathcal{I}_\mathcal{C}}) \in \rho(S, \geq \gamma)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow (a^{\mathcal{I}_\mathcal{C}}, a^{\mathcal{I}_\mathcal{C}}) \in \rho(\exists S.Self, \geq \gamma)^{\mathcal{I}_\mathcal{C}}$. The case $(\exists S.Self)^{\mathcal{I}}(a^I) \leq \gamma$ is similar.
- *Cut concept.* The case is similar to that of fuzzy roles. □

Finally, we are ready now to prove Theorem 3.

Proof of Theorem 3. We will include the proof for the only-if direction, as the converse can easily be obtained by using similar arguments. Since $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ satisfying every axiom in $\mathcal{K}$. Now, it is possible to build the crisp interpretation $\mathcal{I}_{\mathcal{C}}$ that satisfies every axiom in $\texttt{crisp}(\mathcal{K})$. For every axiom $\tau \in \mathcal{K}$, there are several cases:

- τ is a concept assertion. Assume that $\mathcal{I} \models \langle a : C \geq \gamma \rangle$. Then, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma$. By Proposition 11, $(\rho(C, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}(a^{\mathcal{I}_{\mathcal{C}}}) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \geq \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \geq \gamma \rangle)$. The case $\mathcal{I} \models \langle a : C \leq \gamma \rangle$ is similar.
- τ is a role assertion. Assume that $\mathcal{I} \models \langle (a, b) : R \geq \gamma \rangle$. Then, $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \gamma$. By Proposition 10, $(\rho(R, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(R, \geq \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R \geq \gamma \rangle)$. The case $\mathcal{I} \models \langle (a, b) : R \leq \gamma \rangle$ is similar.
- τ is a negated role assertion. The case is similar to the previous one.
- τ is an inequality assertion. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_{\mathcal{C}}} \neq b^{\mathcal{I}_{\mathcal{C}}}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a \neq b \rangle \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a \neq b \rangle)$.
- τ is an equality assertion. The case is similar to the previous one.
- τ is a fuzzy GCI. Assume that $\mathcal{I} \models \langle C \sqsubseteq_F D \geq \gamma \rangle$. Then, $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow_F D^{\mathcal{I}}(x) \geq \gamma$. Hence, for an arbitrary individual $x \in \Delta^{\mathcal{I}}$ it follows that $C^{\mathcal{I}}(x) \Rightarrow_F D^{\mathcal{I}}(x) \geq \gamma$. By Proposition 4, for every pair $(\gamma_x, \gamma_y) \in O^{+}_{\Rightarrow_F \geq \gamma}$, $(C^{\mathcal{I}}(x) \geq \gamma_x$ implies that $D^{\mathcal{I}}(x) \geq \gamma_y$. By induction hypothesis, this implies that for every pair $(\gamma_x, \gamma_y) \in O^{+}_{\Rightarrow_F \geq \gamma}$ and every $x \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, $x \in (\rho(C, \geq \gamma_x)$ implies that $x \in (\rho(D, \geq \gamma_y)$. Consequently, $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle C \sqsubseteq_F D \geq \gamma \rangle)$.
- τ is a fuzzy RIA. The case is similar to the previous one, but now we use Proposition 6.
- τ is a transitive role axiom. Use the equivalence $\texttt{trans}(R) \equiv \langle RR \sqsubseteq_I R \geq \gamma_p \rangle$ for some R-implication I.
- τ is a role disjoint axiom. Assume that $\mathcal{I} \models \texttt{dis}(S_1, S_2)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = \gamma_0$ or $S_2^{\mathcal{I}}(x, y) = \gamma_0$. By Proposition 10, $\forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, $(x, y) \in (\rho(S_1, \leq \gamma_0))^{\mathcal{I}_{\mathcal{C}}}$ or $(x, y) \in (\rho(S_2, \leq \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, y) \notin (\rho(S_1, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}}$ or $(x, y) \notin (\rho(S_2, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow (\rho(S_1, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \cap (\rho(S_2, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}} = \emptyset \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (\texttt{dis}(\rho(S_1, > \gamma_0), \rho(S_2, > \gamma_0))) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\texttt{dis}(S_1, S_2))$.
- τ is a reflexive role axiom. Assume that $\mathcal{I} \models \texttt{ref}(R)$. Then, $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, x) = \gamma_p$. By Proposition 10, $\forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, x) \in (\rho(R, \geq \gamma_p))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, \mathcal{I}_{\mathcal{C}} \models (x, x) : \rho(R, \geq \gamma_p) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\texttt{ref}(R))$.
- τ is an irreflexive role axiom. Assume that $\mathcal{I} \models \texttt{irr}(S)$. Then, $\forall x \in \Delta^{\mathcal{I}}$, $S^{\mathcal{I}}(x, x) = \gamma_0$. By Proposition 10, $\forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, x) \in (\rho(S, \leq \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, x) \notin (\rho(S, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \texttt{irr}(\rho(S, > \gamma_0)) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\texttt{irr}(S))$.
- τ is a symmetry role axiom. Use the equivalence $\texttt{sym}(R) \equiv \langle R \sqsubseteq_I R^{-} \geq \gamma_p \rangle$ for some R-implication I.
- τ is an asymmetry role axiom. Assume that $\mathcal{I} \models \texttt{asy}(S)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x, y) > \gamma_0$ then $S^{\mathcal{I}}(y, x) = \gamma_0$. By Proposition 10, $\forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, if $(x, y) \in (\rho(S, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}}$ then $(y, x) \in (\rho(S, \leq \gamma_0))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, if $(x, y) \in (\rho(S, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}}$ then $(y, x) \notin (\rho(S, > \gamma_0))^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $\mathcal{I}_{\mathcal{C}} \models \kappa(\texttt{asy}(\rho(S, > \gamma_0)))$. □

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