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# CRISP REPRESENTATIONS AND REASONING FOR FUZZY ONTOLOGIES

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Classical ontologies are not suitable to represent imprecise nor uncertain pieces of information. Fuzzy Description Logics were born to represent the former type of knowledge, but they require an appropriate fuzzy language to be agreed on and an important number of available resources to be adapted. This paper faces these problems by presenting a reasoning preserving procedure to obtain a crisp representation for a fuzzy extension of the logic $\mathcal{SROIQ}(\mathbf{D})$ which includes fuzzy nominals and trapezoidal membership functions, and uses Gödel implication in the semantics of fuzzy concept and role subsumption. This reduction makes it possible to reuse a crisp representation language as well as currently available reasoners. Our procedure is optimized with respect to related work, reducing the size of the resulting knowledge base. Finally, we also suggest some further optimizations before applying crisp reasoning.

*Keywords*: Fuzzy description logics; fuzzy ontologies; fuzzy logic; semantic web.

## 1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning.[1] An ontology is defined as an explicit and formal specification of a shared conceptualization,[2] which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies allow to add semantics to data, making knowledge maintenance as well as the reuse of components easier.

The current standard language for ontology creation is the Web Ontology Language (OWL[3]), which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full. OWL Full is the most expressive level but reasoning within it becomes undecidable, OWL Lite has the lowest complexity and OWL DL is a balanced tradeoff between expressiveness and reasoning complexity. However, since its first development, several extensions to OWL have been proposed.[4] Among them, the most significant is OWL 1.1[5] which is its most likely immediate successor.

Description Logics (DLs)[6] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructs of the logic and therefore its complexity. DLs have proved to be very useful as ontology languages.[7] For instance, OWL Lite, OWL DL and OWL 1.1 are close equivalents to $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SROIQ}(\mathbf{D})$ respectively.[8]

Defining a fuzzy DL brings about that the large number of resources available for crisp DLs are no longer appropriate and need to be adapted to the new framework –new fuzzy languages and tools must be developed–, requiring an important effort. This issue affects especially reasoning engines. Previous experiences with crisp DLs have shown that there exists a significant gap between the design of a decision procedure and the achievement of a practical implementation,[11] since expressive DLs has a very high worst-case complexity (e.g. NExpTime in $\mathcal{SHOIN}$). Therefore, optimization of fuzzy DL reasoners will be presumably very hard and costly, which is avoided with our proposal.

An alternative is to represent fuzzy DLs using crisp DLs and to reduce reasoning with fuzzy DLs to reasoning with crisp ones. This has several advantages:

- There is no need to agree on a new standard fuzzy language, but every developer could use its own language expressing fuzzy $\mathcal{SROIQ}(\mathbf{D})$, as long as he implements the reduction that we describe.
- We can continue using standard languages with a lot of resources available. Although it would be desirable to assist the user in tasks such as fuzzy ontology editing, reducing the fuzzy ontology into a crisp one or fuzzy querying, once the reduction is performed, we may use the resources available for the crisp language.
- We may continue using existing crisp reasoners. We do not claim that reasoning will be more efficient, but this approach offers a workaround to support early reasoning in future fuzzy languages. In fact, nowadays there is no reasoner fully supporting a fuzzy extension of OWL 1.1.

Under this approach an immediate practical application of fuzzy ontologies is feasible, because of its tight relation with already existing languages and tools which have proved their validity.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory,[10] the representation of them using crisp description logics has not received such attention. The first effort in this direction is due to U. Straccia, who considered fuzzy $\mathcal{ALCH}$.[12] F. Bobillo *et al.* widened his work to $\mathcal{SHOIN}$.[13] Afterwards, G. Stoilos et al. extended this work with some role constructors.[14] Finally, F. Bobillo *et al.* extended this work with the additional constructors of

$\mathcal{SROIQ}$.[15] For more details of the related work, we refer the reader to Sec. 5.

The contributions of this paper can be summarized as follows:

- We augment the expressivity of fuzzy DLs by allowing the definition of fuzzy sets by extension (fuzzy nominals) and by allowing fuzzy General Concept Inclusions (GCIs) and fuzzy Role Inclusion Axioms (RIAs) to be verified up to some degree, under a novel semantics which uses Gödel implication. We also analyze the properties of the resulting logic.
- We provide a representation of fuzzy $\mathcal{SROIQ}(\mathbf{D})$. Our work includes the extension of the reduction from $\mathcal{ALCH}$ to $\mathcal{SHOIN}$ and from $\mathcal{SROIN}$ to $\mathcal{SROIQ}(\mathbf{D})$. Furthermore, we allow to represent fuzzy concrete predicates, fuzzy GCIs and fuzzy RIAs by using their crisp versions. We also discuss how to allow some atomic concepts and roles to be crisp.
- We optimize the reduction, improving the seminal work by U. Straccia[12] by reducing the number of new atomic elements generated in the reduction as well as the number of axioms. We also show how to optimize some important particular cases of GCIs, how to allow some concepts and roles to be interpreted as crisp, and how to simplify the resulting (crisp) knowledge base before performing reasoning.

The remainder of this paper is organized as follows. The following section reviews some background on DLs and fuzzy logic. Next, Sec. 3 describes a fuzzy extension of $\mathcal{SROIQ}(\mathbf{D})$ and discusses some logical properties. Then, Sec. 4 depicts a reduction into crisp $\mathcal{SROIQ}(\mathbf{D})$. Section 5 reviews some related work and, finally, in Sec. 6 we set out some conclusions and ideas for future work.

## 2. Preliminaries

This section provides some basic background. Section 2.1 quickly overviews $\mathcal{SROIQ}(\mathbf{D})$,[16] the DL which will be mainly treated throughout this paper. Section 2.2 refreshes some basic ideas in fuzzy set theory and fuzzy logic.[17,18]

### 2.1. *The description logic* $\mathcal{SROIQ}(\mathbf{D})$

**Syntax.** A *concrete domain* is a pair $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Delta_{\mathbf{D}}$ is a concrete interpretation domain and $\Phi_{\mathbf{D}}$ is a set of domain predicates $\mathbf{d}$ with a predefined arity $n$ and an interpretation $\mathbf{d}_{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}^n$. For simplicity we assume arity 1.

$\mathcal{SROIQ}(\mathbf{D})$ assumes three alphabets of symbols, for concepts, roles and individuals. The concepts (denoted $C$ or $D$) and abstract roles ($R$) of the language can be built inductively from atomic concepts ($A$), top concept $\top$, bottom concept $\bot$, named individuals ($o_i$), simple roles ($S$, which will be defined below), atomic abstract roles ($R_A$), universal role ($U$) and concrete roles ($T$), as shown in Table 1, where $n, m$ are natural numbers ($n \geq 0, m > 0$), $x, y \in \Delta^{\mathcal{I}}$ are abstract individuals, $v \in \Delta_{\mathbf{D}}$ is a concrete individual and $|X|$ denotes the cardinality of the set $X$.

A Knowledge Base (KB) comprises two parts: the intensional knowledge, i.e., general knowledge about the application domain (a Terminological Box or *TBox* $\mathcal{T}$ and a Role Box or *RBox* $\mathcal{R}$), and the extensional knowledge, i.e., particular knowledge about some specific situation (an Assertional Box or *ABox* $\mathcal{A}$ with statements about individuals).

An ABox consists of a finite set of *assertions* about individuals:

- *concept assertions* $a : C$, meaning that individual $a$ is an instance of $C$,
- *abstract role assertions* $(a, b) : R$, meaning that $(a, b)$ is an instance of $R$, and $(a, b) : \neg R$, meaning that $(a, b)$ is not an instance of $R$,
- *concrete role assertions* $(a, v) : T$ and $(a, v) : \neg T$,
- *inequality assertions* $a \neq b$,
- *equality assertions* $a = b$.

Assertions of the form $(a, b) : \neg R, (a, v) : \neg T$ are called *negated role assertions*.

A TBox consists of a finite set of *general concept inclusion* (*GCI*) axioms $C \sqsubseteq D$ ($C$ is more specific than $D$).

Let $w$ be a role chain (a finite string of roles not including the universal role $U$). An RBox consists of a finite set of role axioms:

- *role inclusion axioms (RIAs)* $w \sqsubseteq R$ ($w$ is more specific than $R$),
- *transitive role axioms* $trans(R)$,
- *disjoint role axioms* $dis(S_1, S_2)$,
- *reflexive role axioms* $ref(R)$,
- *irreflexive role axioms* $irr(S)$,
- *symmetric role axioms* $sym(R)$,
- *asymmetric role axioms* $asy(S)$.

A strict partial order $\prec$ on a set $A$ is an irreflexive and transitive relation on $A$. A strict partial order $\prec$ on the set of roles is called a regular order if it also satisfies $R_1 \prec R_2 \Leftrightarrow R_2^- \prec R_1$, for all roles $R_1$ and $R_2$.

Role axioms cannot contain $U$ and every RIA should be $\prec$-regular, for a regular order $\prec$. A RIA $w \sqsubseteq R$ is $\prec$-*regular* if $R = R_A$ and:

(1) $w = RR$, or
(2) $w = R^-$, or
(3) $w = S_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
(4) $w = RS_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
(5) $w = S_1 \ldots S_n R$ and $S_i \prec R$ for all $i = 1, \ldots, n$.

Note that, in order to prove decidability of the reasoning, roles are assumed to be simple in some concept constructs (local reflexivity, at-least and at-most number restrictions) and role axioms (disjoint, irreflexive and asymmetric role axioms).[16]

*Simple* roles are inductively defined as follows: *(i)* $R_A$ is simple if it does not occur on the right side of a RIA, *(ii)* $R^-$ is simple if $R$ is, and *(iii)* if $R$ occurs on the right side of a RIA, $R$ is simple if, for each $w \sqsubseteq R$, $w = S$ for a simple role $S$.

**Semantics.** An interpretation $\mathcal{I}$ with respect to a concrete domain $\mathbf{D}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and an interpretation function $\cdot^{\mathcal{I}}$ mapping:

- every abstract individual onto an element of $\Delta^{\mathcal{I}}$,
- every concrete individual onto an element of $\Delta_{\mathbf{D}}$,
- every atomic $A$ onto a function $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$,
- every abstract atomic role $R_A$ onto a function $R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$,
- every concrete role $T$ onto a function $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$,
- every $n$-ary concrete predicate $\mathbf{d}$ onto the interpretation $d_{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}^n$.

The interpretation is extended to complex concepts and roles by the inductive definitions in Table 1. Unique name assumption is not imposed, i.e., two nominals might refer to the same individual.

Table 1. Syntax and semantics of the description logic $\mathcal{SROIQ}(\mathbf{D})$.

| Constructor | Syntax | Semantics |
|---|---|---|
| (atomic concept) | $A$ | $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ |
| (top concept) | $\top$ | $\Delta^{\mathcal{I}}$ |
| (bottom concept) | $\bot$ | $\emptyset$ |
| (concept conjunction) | $C \sqcap D$ | $C^{\mathcal{I}} \cap D^{\mathcal{I}}$ |
| (concept disjunction) | $C \sqcup D$ | $C^{\mathcal{I}} \cup D^{\mathcal{I}}$ |
| (concept negation) | $\neg C$ | $\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ |
| (universal quantification) | $\forall R.C$ | $\{x : \forall y, (x,y) \notin R^{\mathcal{I}}$ or $y \in C^{\mathcal{I}}\}$ |
| (existential quantification) | $\exists R.C$ | $\{x : \exists y, (x,y) \in R^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$ |
| (concrete universal quantification) | $\forall T.\mathbf{d}$ | $\{x : \forall v, (x,v) \notin T^{\mathcal{I}}$ or $v \in \mathbf{d_D}\}$ |
| (concrete existential quantification) | $\exists T.\mathbf{d}$ | $\{x : \exists v, (x,v) \in T^{\mathcal{I}}$ and $v \in \mathbf{d_D}\}$ |
| (nominals) | $\{o_1, \ldots, o_m\}$ | $\{o_1^{\mathcal{I}}, \ldots, o_m^{\mathcal{I}}\}$ |
| (at-least number restriction) | $\geq n\ S.C$ | $\{x : \lvert\{y : (x,y) \in S^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}\rvert \geq n\}$ |
| (at-most number restriction) | $\leq n\ S.C$ | $\{x : \lvert\{y : (x,y) \in S^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}\rvert \leq n\}$ |
| (concrete at-least number restriction) | $\geq n\ T.\mathbf{d}$ | $\{x : \lvert\{v : (x,v) \in T^{\mathcal{I}}$ and $v \in \mathbf{d_D}\}\rvert \geq n\}$ |
| (concrete at-most number restriction) | $\leq n\ T.\mathbf{d}$ | $\{x : \lvert\{v : (x,v) \in T^{\mathcal{I}}$ and $v \in \mathbf{d_D}\}\rvert \leq n\}$ |
| (local reflexivity) | $\exists S.Self$ | $\{x : (x,x) \in S^{\mathcal{I}}\}$ |
| (atomic role) | $R_A$ | $R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ |
| (inverse role) | $R^-$ | $\{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\}$ |
| (universal role) | $U$ | $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ |
| (concrete role) | $T$ | $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$ |

An interpretation $\mathcal{I}$ satisfies (is a model of):

- $a : C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $(a,b) : R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$,
- $(a,b) : \neg R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$,

- $(a, v) : T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \in T^{\mathcal{I}}$,
- $(a, v) : \neg T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \notin T^{\mathcal{I}}$,
- $a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $R_1 \dots R_n \sqsubseteq R$ iff $R_1^{\mathcal{I}} \circ \dots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$,
- $trans(R)$ iff $(x, y) \in R^{\mathcal{I}}$ and $(y, z) \in R^{\mathcal{I}}$ imply $(x, z) \in R^{\mathcal{I}}$, $\forall x, y, z \in \Delta^{\mathcal{I}}$,
- $dis(S_1, S_2)$ iff $S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$,
- $ref(R)$ iff $(x, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $irr(S)$ iff $(x, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $sym(R)$ iff $(x, y) \in R^{\mathcal{I}}$ implies $(y, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $asy(S)$ iff $(x, y) \in S^{\mathcal{I}}$ implies $(y, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- a KB $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

A DL not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability, concept satisfiability or subsumption. However, if a DL is closed under negation, all the basic reasoning tasks are reducible to KB satisfiability,[19] so it is usually the only task considered.

### 2.2. *Fuzzy set theory*

Fuzzy set theory and fuzzy logic were proposed by L. Zadeh[20] to manage imprecise and vague knowledge. While in classical set theory elements either belong to a set or not, in fuzzy set theory elements can belong to a set to some degree. More formally, let $X$ be a set of elements called the reference set. A fuzzy subset $A$ of $X$, is defined by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns any $x \in X$ to a value in the interval of real numbers between 0 and 1. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which $x$ can be considered an element of $X$.

The *support* of a fuzzy set $A$ is the set of elements such that their membership degree to $A$ is non-zero i.e., $supp(A) = \{x \mid \mu_A(x) > 0\}$. The *core* of a fuzzy set $A$ is the set of elements which fully belong to $A$ i.e., $core(A) = \{x \mid \mu_A(x) = 1\}$. For every $\alpha \in [0, 1]$, the *$\alpha$-cut* of a fuzzy set $A$ is defined as the (crisp) set such that its elements belong to $A$ with degree at least $\alpha$, i.e., $\{x \mid \mu_A(x) \geq \alpha\}$. Similarly, the *strict $\alpha$-cut* is defined as $\{x \mid \mu_A(x) > \alpha\}$. Notice that these four sets are crisp.

All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication set operations are performed by a t-norm function, a t-conorm function, a negation function and an implication function, respectively. Table 2 shows the most important families of fuzzy operators: Zadeh, Łukasiewicz, Gödel and product. The operators of the Zadeh logic can be represented using the operators of Łukasiewicz logic, so the latter is more general.

The operation of fuzzy intersection is performed by a *t-norm* function $t$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$, i.e., a function satisfying the following properties: *(i) boundary condition* i.e., $t(a, 1) = a$, *(ii) increasing monotonicity* i.e., for each $b \leq c$

Table 2. Popular families of fuzzy operators.

| Family | t-norm $\alpha \otimes \beta$ | t-conorm $\alpha \oplus \beta$ | Negation $\ominus\alpha$ | Implication $\alpha \Rightarrow \beta$ |
|---|---|---|---|---|
| Zadeh | $\min\{\alpha, \beta\}$ | $\max\{\alpha, \beta\}$ | $1 - \alpha$ | $\max\{1 - \alpha, \beta\}$ |
| Łukasiewicz | $\max\{\alpha + \beta - 1, 0\}$ | $\min\{\alpha + \beta, 1\}$ | $1 - \alpha$ | $\min\{1 - \alpha + \beta, 1\}$ |
| Gödel | $\min\{\alpha, \beta\}$ | $\max\{\alpha, \beta\}$ | $\begin{cases} 1, \ \alpha = 0 \\ 0, \ \alpha > 0 \end{cases}$ | $\begin{cases} 1 \ \ \alpha \leq \beta \\ \beta, \ \alpha > \beta \end{cases}$ |
| Product | $\alpha \cdot \beta$ | $\alpha + \beta - \alpha \cdot \beta$ | $\begin{cases} 1, \ \alpha = 0 \\ 0, \ \alpha > 0 \end{cases}$ | $\begin{cases} 1 \quad \alpha \leq \beta \\ \beta/\alpha, \ \alpha > \beta \end{cases}$ |

then $t(a, b) \leq t(a, c)$, *(iii) commutativity* i.e., $t(a, b) = t(b, a)$, *(iv) associativity* i.e., $t(a, t(b, c)) = t(t(a, b), c)$. Every t-norm $t$ satisfies $a, b \geq t(a, b)$ and $t(a, 0) = 0$.

Fuzzy union is performed by a *t-conorm* (or s-norm) function $u : [0, 1] \times [0, 1] \rightarrow [0, 1]$, i.e., a function satisfying: *(i) boundary condition* i.e., $u(a, 0) = a$, *(ii) increasing monotonicity* i.e., for each $b \leq c$ then $u(a, b) \leq u(a, c)$, *(iii) commutativity* i.e., $u(a, b) = u(b, a)$, *(iv) associativity* i.e., $u(a, u(b, c)) = u(u(a, b), c)$. Every t-conorm $u$ satisfies $a, b \leq u(a, b)$, and $u(a, 1) = 1$.

Fuzzy complement is performed by a *negation* function $c : [0, 1] \rightarrow [0, 1]$ satisfying: (i) *boundary conditions* i.e., $c(0) = 1$ and $c(1) = 0$, *(ii) decreasing monotonicity* i.e., for each $a \leq b, c(a) \geq c(b)$. Łukasiewicz negation additionally satisfies *continuity* and *involution* i.e., for each $a \in [0, 1]$ $c(c(a)) = a$ holds.

Fuzzy implication is performed by an *implication* function $i : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following properties: *(i) monotonicity* i.e., $a \leq b \Rightarrow i(a, c) \geq i(b, c)$ and $b \leq c \Rightarrow i(a, b) \leq i(a, c)$, *(ii) boundary conditions* i.e., $i(0, a) = i(a, 1) = 1$ and $i(1, 0) = 0$. There are commonly two types of fuzzy implications used. The first class is *S-implications*, which extend the crisp proposition $a \Rightarrow b = \neg a \vee b$ to the fuzzy case and are defined by the operation $i(a, b) = u(c(a), b)$. The second class is *R-implications* (residuum-based implications), which are defined as $i(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}$ and can be used to define a fuzzy complement as $c(a) = i(a, 0)$. They always verify that $i(a, b) = 1$ iff $a \leq b$. Product and Gödel implications are R-implications, the implication of the Zadeh family, which is called Kleene-Dienes (KD), is an S-implication and the Łukasiewicz implication belongs to both types.

## 3. Fuzzy $\mathcal{SROIQ}(\mathbf{D})$

In this section we define $f\mathcal{SROIQ}(\mathbf{D})$, which extends $\mathcal{SROIQ}(\mathbf{D})$ to the fuzzy case by letting concepts denote fuzzy sets of individuals and roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. The following definition combines previous works,[14,21] with the novelty of using fuzzy nominals and Gödel implication in the semantics of GCIs and RIAs.

### 3.1. *Definition*

**Syntax.** A *fuzzy concrete domain* $\mathbf{D}$ is a pair $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Delta_{\mathbf{D}}$ is a concrete interpretation domain and $\Phi_{\mathbf{D}}$ is a set of fuzzy domain predicates $\mathbf{d}$ with a predefined arity $n$ and an interpretation $\mathbf{d}_{\mathbf{D}} : \Delta_{\mathbf{D}}^n \to [0,1]$, which is a $n$-ary fuzzy relation over $\Delta_{\mathbf{D}}$.[22] We will restrict ourselves to the trapezoidal membership function $trap : \mathbb{Q} \cap [k_1, k_2] \to [0,1]$ which is defined as follows:

(1) $trap_{k_1,k_2}(x; a, b, c, d) = (x-a)/(b-a)$, if $x \in [a, b]$
(2) $trap_{k_1,k_2}(x; a, b, c, d) = 1$, if $x \in [b, c]$
(3) $trap_{k_1,k_2}(x; a, b, c, d) = (d-x)/(d-c)$, if $x \in [c, d]$
(4) $trap_{k_1,k_2}(x; a, b, c, d) = 0$ if $x \in [k_1, a] \cup [d, k_2]$

We use trapezoidal membership function because it can be used to represent other popular membership functions such as the triangular $tri_{k_1,k_2}(x; a, b, c)$, the left shoulder function $L_{k_1,k_2}(x; a, b)$ and the right shoulder function $R_{k_1,k_2}(x; a, b)$ (other functions usually considered in fuzzy concrete domains[22]) as $trap_{k_1,k_2}(x; a, b, b, c)$, $trap_{k_1,k_2}(x; k_1, k_1, a, b)$ and $trap_{k_1,k_2}(x; a, b, k_2, k_2)$ respectively.

$f\mathcal{SROIQ}(\mathbf{D})$ assumes three alphabets of symbols, for concepts, roles and individuals. Let $n, m$ be natural numbers ($n \geq 0, m > 0$) and $\alpha_i \in (0,1]$. The concepts (denoted $C$ or $D$) and abstract roles (denoted $R$) of the language can be built inductively from atomic concepts ($A$), top concept $\top$, bottom concept $\bot$, named individuals ($o_i$), simple roles ($S$, which will be defined below), atomic roles ($R_A$), universal role ($U$) and concrete roles ($T$) as shown in Table 3. The only difference with the crisp case are fuzzy nominals.[13]

In the rest of the paper we will assume $\bowtie \in \{\geq, <, \leq, >\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$ and $\gamma \in [0,1]$. The symmetric $\bowtie^-$ and the negation $\neg \bowtie$ of an operator $\bowtie$ are defined as:

| $\bowtie$ | $\bowtie^-$ | $\neg \bowtie$ |
|---|---|---|
| $\geq$ | $\leq$ | $<$ |
| $>$ | $<$ | $\leq$ |
| $\leq$ | $\geq$ | $>$ |
| $<$ | $>$ | $\geq$ |

A fuzzy Knowledge Base (fKB) comprises a fuzzy ABox $\mathcal{A}$ , a fuzzy Terminological Box $\mathcal{T}$ and a fuzzy RBox $\mathcal{R}$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions.* A fuzzy assertion can be an inequality assertion $\langle a \neq b \rangle$, an equality assertion $\langle a = b \rangle$ or a constraint on the truth value of a concept or role assertion, i.e., an expression of the form $\langle \Psi \geq \alpha \rangle$, $\langle \Psi > \beta \rangle$, $\langle \Psi \leq \beta \rangle$ or $\langle \Psi < \alpha \rangle$, where $\Psi$ is of the form $a{:}C$, $(a,b){:}R$ or $(a,v){:}T$.

A *fuzzy TBox* consists of *fuzzy GCIs*, which constrain the truth value of a GCI i.e., they are expressions of the form $\langle \Omega \geq \alpha \rangle$ or $\langle \Omega > \beta \rangle$, where $\Omega = C \sqsubseteq D$.

A *fuzzy RBox* consists of a finite set of role axioms, which can be *fuzzy RIAs* $\langle w \sqsubseteq R \geq \alpha \rangle$ or $\langle w \sqsubseteq R > \beta \rangle$ for a role chain $w = R_1 R_2 \ldots R_n$, or any other of the role axioms from the crisp case: *transitive* $trans(R)$, *disjoint* $dis(S_1, S_2)$, *reflexive* $ref(R)$, *irreflexive* $irr(S)$, *symmetric* $sym(R)$ or *asymmetric* $asy(S)$.

A fuzzy axiom is *positive* (denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geq \alpha \rangle$ or $\langle \tau > \beta \rangle$, and *negative* (denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leq \beta \rangle$ or $\langle \tau < \alpha \rangle$. $\langle \tau = \alpha \rangle$ is equivalent to the pair of axioms $\langle \tau \geq \alpha \rangle$ and $\langle \tau \leq \alpha \rangle$.[23] Of course, $\tau \equiv \langle \tau \geq 1 \rangle$.

As in the crisp case, role axioms cannot contain $U$ and every RIA should be $\prec$-regular for a regular order $\prec$. A RIA $\langle w \sqsubseteq R \rhd \gamma \rangle$ is *$\prec$-regular* if $R = R_A$ and: *(i)* $w = RR$, or *(ii)* $w = R^-$, or *(iii)* $w = S_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or *(iv)* $w = RS_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or *(v)* $w = S_1 \dots S_n R$ and $S_i \prec R$ for all $i = 1, \dots, n$.

*Simple* roles are defined as in the crisp case: *(i)* $R_A$ is simple if does not occur on the right side of a RIA, *(ii)* $R^-$ is simple if $R$ is, *(iii)* if $R$ occurs on the right side of a RIA, $R$ is simple if, for each $\langle w \sqsubseteq R \rhd \gamma \rangle$, $w = S$ for a simple role $S$.

Notice that negative GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs respectively, which are not part of crisp $\mathcal{SROIQ}(\mathbf{D})$.

**Semantics.** A fuzzy interpretation $\mathcal{I}$ with respect to a fuzzy concrete domain $\mathbf{D}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

- every abstract individual $a$ onto an element of $\Delta^{\mathcal{I}}$,
- every concrete individual $v$ onto an element of $\Delta_{\mathbf{D}}$,
- every concept $C$ onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$,
- every abstract role $R$ onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$,
- every concrete role $T$ onto a function $T^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \rightarrow [0, 1]$,
- every $n$-ary concrete fuzzy predicate $\mathbf{d}$ onto the fuzzy relation $d_{\mathbf{D}} : \Delta^n_{\mathbf{D}} \rightarrow [0, 1]$.

$C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) denotes the membership function of the fuzzy concept $C$ (resp. fuzzy role $R$) w.r.t. $\mathcal{I}$. $C^{\mathcal{I}}(x)$ (resp. $R^{\mathcal{I}}(x, y)$) gives us the degree of being the individual $x$ an element of the fuzzy concept $C$ (resp. the degree of being $(x, y)$ an element of the fuzzy role $R$) under the fuzzy interpretation $\mathcal{I}$. We do not impose unique name assumption, i.e., two nominals might refer to the same individual. Given a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- $\langle a : C \bowtie \gamma \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$,
- $\langle (a, b) : R \bowtie \gamma \rangle$ iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \gamma$,
- $\langle (a, v) : T \bowtie \gamma \rangle$ iff $T^{\mathcal{I}}(a^{\mathcal{I}}, v_{\mathbf{D}}) \bowtie \gamma$,
- $\langle a \neq b \rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $\langle a = b \rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $\langle C \sqsubseteq D \rhd \gamma \rangle$ iff $\inf_{x \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \} \rhd \gamma$,
- $\langle R_1 \dots R_n \sqsubseteq R \rhd \gamma \rangle$ iff $\sup_{x_1 \dots x_{n+1} \in \Delta^{\mathcal{I}}} \bigotimes [R_1^{\mathcal{I}}(x_1, x_2), \dots, R_n^{\mathcal{I}}(x_n, x_{n+1})] \Rightarrow R^{\mathcal{I}}(x_1, x_{n+1}) \rhd \gamma$,

Table 3. Syntax and semantics of fuzzy $\mathcal{SROIQ}(\mathbf{D})$.

| Syntax ($C$) | Semantics ($C^{\mathcal{I}}(x)$) |
|---|---|
| $\top$ | $1$ |
| $\bot$ | $0$ |
| $A$ | $A^{\mathcal{I}}(x)$ |
| $C \sqcap D$ | $C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$ |
| $C \sqcup D$ | $C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$ |
| $\neg C$ | $\ominus C^{\mathcal{I}}(x)$ |
| $\forall R.C$ | $\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$ |
| $\exists R.C$ | $\sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$ |
| $\forall T.\mathbf{d}$ | $\inf_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \Rightarrow \mathbf{d}_{\mathbf{D}}(v)\}$ |
| $\exists T.\mathbf{d}$ | $\sup_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \otimes \mathbf{d}_{\mathbf{D}}(v)\}$ |
| $\{\alpha_1/o_1, \dots, \alpha_m/o_m\}$ | $\sup_{i \mid x = o_i^{\mathcal{I}}} \alpha_i$ |
| $\geq m\ S.C$ | $\sup_{y_1,\dots,y_m \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \bigotimes (\otimes_{j<k} \{y_j \neq y_k\})]$ |
| $\leq n\ S.C$ | $\inf_{y_1,\dots,y_{n+1} \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})]$ |
| $\geq m\ T.\mathbf{d}$ | $\sup_{v_1,\dots,v_m \in \Delta_{\mathbf{D}}} [(\otimes_{i=1}^{n} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \bigotimes (\otimes_{j<k} \{v_j \neq v_k\})]$ |
| $\leq n\ T.\mathbf{d}$ | $\inf_{v_1,\dots,v_{n+1} \in \Delta_{\mathbf{D}}} [(\otimes_{i=1}^{n+1} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \Rightarrow (\oplus_{j<k} \{v_j = v_k\})]$ |
| $\exists S.Self$ | $S^{\mathcal{I}}(x,x)$ |
| $R_A$ | $R_A^{\mathcal{I}}(x,y)$ |
| $R^-$ | $R^{\mathcal{I}}(y,x)$ |
| $U$ | $1$ |
| $T$ | $T^{\mathcal{I}}(x,v)$ |

- $trans(R)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$,
- $dis(S_1, S_2)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = 0$ or $S_2^{\mathcal{I}}(x,y) = 0$,
- $ref(R)$ iff $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$,
- $irr(S)$ iff $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$,
- $sym(R)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$,
- $asy(S)$ iff $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$,
- a fKB iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

Notice that individual assertions are considered to be crisp, since the equality and inequality of individuals has always been considered crisp in the fuzzy DL literature.[21,24]

In the rest of the paper we will only consider fKB satisfiability, since (as in the crisp case) most inference problems can be reduced to it.[25] Some examples are:

- *Concept satisfiability.* $C$ is $\alpha$-satisfiable w.r.t. a fKB $f$ iff $f \cup \{\langle x : C \geq \alpha \rangle\}$ is satisfiable, where $x$ is a new individual, which does not appear in $f$.
- *Entailment*: A fuzzy concept assertion $a : C \bowtie \alpha$ is entailed by a fKB $f$ (denoted $f \models \langle a : C \bowtie \alpha \rangle$) iff $f \cup \{\langle a : C \neg\bowtie \alpha \rangle\}$ is unsatisfiable. The case for fuzzy role assertions is similar.

- *Greatest lower bound.* The greatest lower bound of a concept or role assertion $\tau$ is defined as the $\sup\{\alpha : f \models \langle \tau \geq \alpha \rangle\}$ and it can computed by performing several entailment tests[a].

We will shortly justify our decision of fuzzifying the nominal construct — one of the original contributions of this paper — by showing an example.

**Example 1.** Suppose we want to represent the concept of country where German is a widely spoken language. Previous approaches allow to represent a fuzzy disjunction of nominals $C \equiv \{germany\} \sqcup \{austria\} \sqcup \{switzerland\}$. Since the semantics of the nominal construct is crisp ($\{o_i\}^{\mathcal{I}}(x) = 1$ if $x = o_i^{\mathcal{I}}$ or 0 otherwise), it forces *switzerland* to fully belong to the concept or not, despite of German-speaking community of Switzerland represents only about two thirds of the total population of the country. On the contrary, following our approach we are able to define $C \equiv \{1/germany, 1/austria, 0.67/switzerland\}$. □

Let us comment the semantics of the fuzzy nominals $\{\alpha_1/o_1, \dots, \alpha_m/o_m\}^{\mathcal{I}}(x) = \sup_{i \mid x = o_i^{\mathcal{I}}} \alpha_i$. Since we are not imposing unique name assumption, it is possible that $x = o_i^{\mathcal{I}}$ for more than one $o_i$. This is the reason why we need to compute the supremum over the $\alpha_i$ associated to these named individuals $o_i$. And, of course, if $\forall i \in \{1, \dots, m\}, x \neq o_i^{\mathcal{I}}$, then $\{\alpha_1/o_1, \dots, \alpha_m/o_m\}^{\mathcal{I}}(x) = \sup \emptyset = 0$.

Note that previous approaches consider nominals to be crisp singletons arguing that they do not represent real-life concepts.[21,26] In these approaches it is possible to represent a fuzzy disjunction of crisp singletons. However, we consider fuzzy nominals as proper fuzzy sets, which do represent real-life concepts. It is easy to see that our definition generalizes the previous definition for the nominal construct, as $\{o_1\} \sqcup \cdots \sqcup \{o_m\}$ is equivalent to $\{1/o_1, \dots, 1/o_m\}$.

Sometimes it is possible to represent explicitly the vagueness of a concept by defining a fuzzy concrete domain,[22] for example, a trapezoidal membership function defined over the rational numbers. However, sometimes there exist concepts without a subjacent semantic representation, so it is not possible or unnatural to define a fuzzy concrete domain (for example, the concept in Example 1). In these cases, fuzzy nominals allow to explicitly define the membership function of a fuzzy set, stating the meaning that a fuzzy concept has for the ontology developer.

### 3.2. *Logical properties*

It can be easily shown that $f\mathcal{SROIQ}(\mathbf{D})$ is a sound extension of crisp $\mathcal{SROIQ}(\mathbf{D})$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{0, 1\}$.

In the fuzzy DLs literature, the notation $f_i\mathcal{DL}$ has been proposed,[26] where $i$ is the fuzzy implication function considered. Hereinafter we will concentrate on

[a]More precisely, at most $log|N^{fK}|$ tests, where $N^{fK}$ is a set of degrees as defined in Sec. 4.

$f_{KD}\mathcal{SROIQ}(\mathbf{D})$, restricting ourselves to the Zadeh family: minimum t-norm, maximum t-conorm, Łukasiewicz negation and KD implication, with the exception of GCIs and RIAs, where we will consider Gödel implication. This choice comes from the fact that KD implication specifies a t-norm, a t-conorm and a negation which make possible the reduction to a crisp KB, as we will see in Sec. 4 (in principle, other fuzzy operators are not suitable for a similar reduction). However, the use of KD implication in the semantics of GCIs and RIAs brings about two counter-intuitive effects (see Sec. 5 for details), so we will use Gödel implication which is also suitable for a classical representation as we will see in Sec. 4.

In general, Gödel implication provides better logical properties than KD, but KD for example allows to reason with *modus tolens*.[13] A representation language could allow the use of two types of GCIs and RIAs $\sqsubseteq_{KD}$ y $\sqsubseteq_{G}$ (with semantics based on KD and Gödel implications respectively), similarly as other works which allow several types of subsumption.[27] This way, the ontology developer would be free to choose the better option for his own needs. F. Bobillo *et al.* showed how to reduce fuzzy GCIs under KD semantics, and fuzzy RIAs can be reduced similarly.[13]

Due to the standard properties of the fuzzy operators, the following concept equivalences hold:[25] $\neg\top \equiv \bot$, $\neg\bot \equiv \top$, $C \sqcap \top \equiv C$, $C \sqcup \bot \equiv C$, $C \sqcap \bot \equiv \bot$, $C \sqcup \top \equiv \top$, $\exists R.\bot = \bot$, $\forall R.\top = \top$. Laws of *excluded middle* and *contradiction* do not hold i.e. $C \sqcup \neg C \not\equiv \top$ and $C \sqcap \neg C \not\equiv \bot$. Moreover, the choice of the fuzzy operators implies that the following properties (which extend the crisp case) hold:

- *Involution*: $\neg\neg C \equiv C$.
- *De Morgan* laws: $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$ and $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$.
- *Inter-definability of quantifiers*: $\forall R.C = \neg\exists R.(\neg C)$ and $\exists R.C = \neg\forall R.(\neg C)$.
- *Inter-definability of cardinality restrictions*: $(\leq n\ S.C) \equiv \neg(\geq n{+}1\ S.C)$ and $(\geq m\ S.C) \equiv \neg(\leq m{-}1\ S.C)$.

It is possible to transform concept expressions into a semantically equivalent *Negation Normal Form* (NNF), which is obtained by pushing in the usual manner negation in front of atomic concepts, fuzzy nominals and local reflexivity concepts.

In crisp DLs, the assertion $a\!:\!C$ is equivalent to the GCI $\{a\} \sqsubseteq C$. This can be extended to the fuzzy case, because $\langle a\!:\!C \geq \alpha\rangle$ is equivalent to $\langle \{\alpha/a\} \sqsubseteq C \geq 1\rangle$.

Similarly as in the original work of U. Straccia,[28] $f_{KD}\mathcal{SROIQ}(\mathbf{D})$ allows some sort of *modus ponens* and *chaining* of GCIs and RIAs:

**Proposition 1.** *For $\alpha, \beta \in [0,1]$ and $\rhd \in \{\geq, >\}$, the following properties hold:*

(i) $\langle a\!:\!C \rhd \alpha\rangle$ *and* $\langle C \sqsubseteq D \rhd \beta\rangle$ *imply* $\langle a\!:\!D \rhd \min\{\alpha,\beta\}\rangle$.
(ii) $\langle (a,b)\!:\!R \rhd \alpha\rangle$ *and* $\langle R \sqsubseteq R' \rhd \beta\rangle$ *imply* $\langle (a,b)\!:\!R' \rhd \min\{\alpha,\beta\}\rangle$.
(iii) $\langle C \sqsubseteq D \rhd \alpha\rangle$ *and* $\langle D \sqsubseteq E \rhd \beta\rangle$ *imply* $\langle C \sqsubseteq E \rhd \min\{\alpha,\beta\}\rangle$.
(iv) $\langle R \sqsubseteq R' \rhd \alpha\rangle$ *and* $\langle R' \sqsubseteq R'' \rhd \beta\rangle$ *imply* $\langle R \sqsubseteq R'' \rhd \min\{\alpha,\beta\}\rangle$.

Irreflexive, transitive and symmetric role axioms are syntactic sugar for any R-implication (and consequently it can be assumed that they do not appear in fKBs) due to the following equivalences:

- $irr(S) \equiv \langle \top \sqsubseteq \neg \exists S.Self \geq 1 \rangle$,
- $trans(R) \equiv \langle RR \sqsubseteq R \geq 1 \rangle$,
- $sym(R) \equiv \langle R \sqsubseteq R^{-} \geq 1 \rangle$.

## 4. An Optimized Crisp Representation for Fuzzy $\mathcal{SROIQ}$(D)

In this section we show how to reduce a $f_{KD}\mathcal{SROIQ}(\mathbf{D})$ fKB into a crisp KB. The procedure preserves reasoning, so existing $\mathcal{SROIQ}(\mathbf{D})$ reasoners could be applied to the resulting KB. First we will describe the reduction and then we will provide an illustrating example.

The basic idea is to create some new crisp concepts and roles, representing the $\alpha$-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

### 4.1. *Adding new elements*

Let $\mathbf{A}$ be the set of atomic concepts, $\mathbf{R}$ the set of atomic abstract roles and $\mathbf{T}$ the set of concrete roles in a fKB $fK = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$. U. Straccia showed[12] that the set of the degrees which must be considered for any reasoning task is defined as

$$N^{fK} = X^{fK} \cup \{1 - \gamma \mid \gamma \in X^{fK}\}$$

where $X^{fK} = \{0, 0.5, 1\} \cup \{\gamma \mid \langle \tau \bowtie \gamma \rangle \in fK\}$. This also holds in $f_{KD}\mathcal{SROIQ}(\mathbf{D})$, but it must be noted that it is not necessarily true when other fuzzy operators are considered.

Without loss of generality, it can be assumed that $N^{fK} = \{\gamma_1, \ldots, \gamma_{|N^{fK}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leq i \leq |N^{fK}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|N^{fK}|} = 1$.

Now, for each $\alpha, \beta \in N^{fK}$ with $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, for each $A \in \mathbf{A}$, two new atomic concepts $A_{\geq \alpha}, A_{> \beta}$. are introduced. $A_{\geq \alpha}$ represents the crisp set of individuals which are instance of $A$ with degree higher or equal than $\alpha$ i.e the $\alpha$-cut of $A$. $A_{> \beta}$ is defined in a similar way. Similarly, for each $R_A \in \mathbf{R}$ and for each $T \in \mathbf{T}$ two new atomic abstract roles $R_{A \geq \alpha}, R_{A > \beta}$ and two concrete roles $T_{\geq \alpha}, T_{> \beta}$ are introduced. The atomic elements $A_{>1}, R_{A>1}, A_{\geq 0}$ and $R_{A \geq 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fKB (e.g. we do not allow GCIs of the form $C \sqsubseteq D \geq 0$). Note that previous works also consider unnecessarily $A_{\geq 0}$ and $R_{A \geq 0}$.[12,13]

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq |N^{fK}| - 1, 2 \leq j \leq |N^{fK}| - 1$ and for each $A \in \mathbf{A}$, $T(N^{fK})$ is the smallest terminology containing these two axioms: $A_{\geq \gamma_{i+1}} \sqsubseteq A_{> \gamma_i}$ and $A_{> \gamma_j} \sqsubseteq A_{\geq \gamma_j}$. Similarly, for each $R_A \in \mathbf{R}$, $R_a(N^{fK})$ is the smallest terminology containing $R_{A \geq \gamma_{i+1}} \sqsubseteq R_{A > \gamma_i}$ and $R_{A > \gamma_i} \sqsubseteq R_{A \geq \gamma_i}$; for each $T \in \mathbf{T}$, $R_c(N^{fK})$ is the smallest terminology containing $T_{\geq \gamma_{i+1}} \sqsubseteq T_{> \gamma_i}$ and $T_{> \gamma_i} \sqsubseteq T_{\geq \gamma_i}$.

Previous works[12,13] use two more atomic concepts $A_{\leq \beta}, A_{<\alpha}$ and the following additional axioms (for $2 \leq k \leq |N^{fK}|$):

$$\begin{array}{rcl} A_{<\gamma_k} & \sqsubseteq & A_{\leq \gamma_k}, \\ A_{\geq \gamma_k} \sqcap A_{<\gamma_k} & \sqsubseteq & \bot, \\ \top & \sqsubseteq & A_{\geq \gamma_k} \sqcup A_{<\gamma_k}, \end{array} \qquad \begin{array}{rcl} A_{\leq \gamma_i} & \sqsubseteq & A_{<\gamma_{i+1}} \\ A_{>\gamma_i} \sqcap A_{\leq \gamma_i} & \sqsubseteq & \bot \\ \top & \sqsubseteq & A_{>\gamma_i} \sqcup A_{\leq \gamma_i} \end{array}$$

In contrast to this, we use $\neg A_{>\gamma_k}$ rather than $A_{\leq \gamma_k}$ and $\neg A_{\geq \gamma_k}$ instead of $A_{<\gamma_k}$. The six axioms above follow immediately from the semantics of the crisp concepts as Proposition 2 shows:

**Proposition 2.** *If $A_{\geq \gamma_{i+1}} \sqsubseteq A_{>\gamma_i}$ and $A_{>\gamma_k} \sqsubseteq A_{\geq \gamma_k}$ hold, then the followings axioms are verified:*

$$\begin{array}{ll} (1)\ \neg A_{\geq \gamma_k} \sqsubseteq \neg A_{>\gamma_k} & (2)\ \neg A_{>\gamma_i} \sqsubseteq \neg A_{\geq \gamma_{i+1}} \\ (3)\ A_{\geq \gamma_k} \sqcap \neg A_{\geq \gamma_k} \sqsubseteq \bot & (4)\ A_{>\gamma_i} \sqcap \neg A_{>\gamma_i} \sqsubseteq \bot \\ (5)\ \top \sqsubseteq A_{\geq \gamma_k} \sqcup \neg A_{\geq \gamma_k} & (6)\ \top \sqsubseteq A_{>\gamma_i} \sqcup \neg A_{>\gamma_i} \end{array}$$

(1) and (2) derive from the fact that in crisp DLs $A \sqsubseteq B \equiv \neg B \sqsubseteq \neg A$. (3) and (4) come from the law of contradiction $A \sqcap \neg A \sqsubseteq \bot$, while (5) and (6) derive from the law of excluded middle $\top \sqsubseteq A \sqcup \neg A$. Moreover, we do not introduce the axiom $A_{>0} \sqsubseteq A_{\geq 0}$; since $A_{\geq 0}$ is equivalent to $\top$ the axiom trivially holds.

In the case of roles, this optimization is essential in order to represent some role constructors of $\mathcal{SROIQ}(\mathbf{D})$ (negated role assertions and self reflexivity concepts). Actually, it is not possible to use a role of the form $R_{A \leq \gamma_k}$ rather than $\neg R_{A>\gamma_k}$ and $R_{A<\gamma_k}$ instead of $\neg R_{A \geq \gamma_k}$ because the logic does not allow to express the corresponding version of the axioms (3), (4), (5) and (6), which would be necessary to guarantee the correctness of the reduction, because the role conjunction and the bottom role are not allowed, and the universal role cannot appear in RIAs.

Now we will introduce some customized datatypes which will be used to represent membership trapezoidal functions. *real*$[a, b]$ denotes a real number defined in $[a, b]$ and can be defined in OWL 1.1 syntax as follows:

```
<owl:DataRange rdf:about="#real[a,b]">
  <owl11:onDataRange rdf:resource="&xsd;double"/>
  <owl11:minInclusive rdf:datatype="&xsd;double">a</owl11:minInclusive>
  <owl11:maxInclusive rdf:datatype="&xsd;double">b</owl11:maxInclusive>
</owl:DataRange>
```

*real*$(a, b)$ denotes a real number defined in $(a, b)$ and can be defined similarly as before, but using `owl11:minExclusive` and `owl11:maxExclusive` instead of `owl11:minInclusive` and `owl11:maxInclusive`.

Finally, *union–real*$[k_1, a, b, k_2]$ stands for a real number in $[k_1, a] \cup [b, k_2]$ and can be defined in the following way:

```
<owl:DataRange rdf:about="#union-real[k1,a,b,k2]">
  <owl:complementOf>
    <owl:DataRange>
      <owl11:onDataRange rdf:resource="real(a,b)"/>
    </owl:DataRange>
  </owl:complementOf>
  <owl11:minInclusive rdf:datatype="&xsd;double">k1</owl11:minInclusive>
  <owl11:maxInclusive rdf:datatype="&xsd;double">k2</owl11:maxInclusive>
</owl:DataRange>
```

Notice that these customized datatypes are just subsets of $\mathbb{R}$, that is, double numbers with a restricted set of allowed values. Hence, it is possible to use current available algorithms to reason with customized real numbers. Observe that using an own datatype representing trapezoidal membership functions is possible but we would need to implement a new reasoning procedure.

### 4.2. *Mapping fuzzy concepts, roles and axioms*

Fuzzy concept, role and concrete predicate expressions are reduced using mapping $\rho$, as shown in Table 4. We recall that we are allowing only fuzzy concrete predicates $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$. Given a fuzzy concept $C$, $\rho(C, \geq \alpha)$ is a crisp set containing all the elements which belong to $C$ with a degree greater or equal than $\alpha$. The other cases $\bowtie \gamma$ are similar. Hence, it holds that $\rho(C, \bowtie \gamma) \equiv \neg\rho(C, \neg \bowtie \gamma)$. $\rho$ is defined in a similar way for fuzzy roles and this equivalence also holds.

Mapping $\rho$ deserves some comments. Firstly, it is interesting to remark that $\rho(A, \leq \beta) = \neg A_{>\beta}$ is different to $\rho(\neg A, \geq \alpha) = \rho(A, \leq 1-\alpha) = \neg A_{>1-\alpha}$. Secondly, in concrete predicates we use a sufficiently small number $\epsilon > 0$ which simulates strict inequalities, since $a \geq b + \epsilon$ is equivalent to $a > b$. Finally, due to the restrictions in the definition of the fKB, some expressions cannot appear during the process:

- $\rho(R, \lhd\gamma)$, $\rho(U, \lhd\gamma)$ and $\rho(T, \lhd\gamma)$ can only appear in a (crisp) negated role assertion.
- $\rho(A, \geq 0), \rho(A, > 1), \rho(A, \leq 1)$ and $\rho(A, < 0)$ cannot appear due to the existing restrictions on the degree of the axioms in the fKB. The same also holds for $\top$, $\perp$ and $R_A$.

Axioms are reduced as in Table 5, where $\kappa(\tau)$ maps a fuzzy axiom $\tau$ in $f_{KD}\mathcal{SROIQ}(D)$ into a set of crisp axioms in $\mathcal{SROIQ}(D)$. We note $\kappa(\mathcal{A})$ (resp. $\kappa(\mathcal{T})$, $\kappa(\mathcal{R})$) the union of the reductions of all the fuzzy axioms in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$)[b]. Observe that $\kappa(\langle C \sqsubseteq D \geq 1\rangle)$ is equivalent to the reduction of a GCI under a semantics based on Zadeh's set inclusion proposed by U. Straccia,[12] although he introduces some unnecessary axioms ($C_{\geq 0} \sqsubseteq D_{\geq 0}$ and $C_{>1} \sqsubseteq D_{>1}$).

[b]More precisely, the reduction of fuzzy GCIs and RIAs should be noted as $\kappa(\tau, N^{fK})$. Similarly, the reduction of the fuzzy TBox and RBox should be noted as $\kappa(\mathcal{T}, N^{fK})$ and $\kappa(\mathcal{R}, N^{fK})$ respectively. However, for the sake of simplicity we omit $N^{fK}$ since it is clear from the context.

Table 4. Mapping of concept, role and concrete predicate expressions.

| $x$ | $y$ | $\rho(x,y)$ |
|---|---|---|
| $\top$ | $\triangleright\gamma$ | $\top$ |
| $\top$ | $\triangleleft\gamma$ | $\bot$ |
| $\bot$ | $\triangleright\gamma$ | $\bot$ |
| $\bot$ | $\triangleleft\gamma$ | $\top$ |
| $A$ | $\triangleright\gamma$ | $A_{\triangleright\gamma}$ |
| $A$ | $\triangleleft\gamma$ | $\neg A_{\neg\triangleleft\gamma}$ |
| $\neg C$ | $\bowtie\gamma$ | $\rho(C, \bowtie^{-} 1-\gamma)$ |
| $C \sqcap D$ | $\triangleright\gamma$ | $\rho(C,\triangleright\gamma) \sqcap \rho(D,\triangleright\gamma)$ |
| $C \sqcap D$ | $\triangleleft\gamma$ | $\rho(C,\triangleleft\gamma) \sqcup \rho(D,\triangleleft\gamma)$ |
| $C \sqcup D$ | $\triangleright\gamma$ | $\rho(C,\triangleright\gamma) \sqcup \rho(D,\triangleright\gamma)$ |
| $C \sqcup D$ | $\triangleleft\gamma$ | $\rho(C,\triangleleft\gamma) \sqcap \rho(D,\triangleleft\gamma)$ |
| $\exists R.C$ | $\triangleright\gamma$ | $\exists\rho(R,\triangleright\gamma).\rho(C,\triangleright\gamma)$ |
| $\exists R.C$ | $\triangleleft\gamma$ | $\forall\rho(R,\neg\triangleleft\gamma).\rho(C,\triangleleft\gamma)$ |
| $\forall R.C$ | $\{\geq,>\}\gamma$ | $\forall\rho(R,\{>,\geq\}1-\gamma).\rho(C,\{\geq,>\}\gamma)$ |
| $\forall R.C$ | $\triangleleft\gamma$ | $\exists\rho(R,\triangleleft^{-}1-\gamma).\rho(C,\triangleleft\gamma)$ |
| $\exists T.\mathbf{d}$ | $\triangleright\gamma$ | $\exists\rho(T,\triangleright\gamma).\rho(\mathbf{d},\triangleright\gamma)$ |
| $\exists T.\mathbf{d}$ | $\triangleleft\gamma$ | $\forall\rho(T,\neg\triangleleft\gamma).\rho(\mathbf{d},\triangleleft\gamma)$ |
| $\forall T.\mathbf{d}$ | $\{\geq,>\}\gamma$ | $\forall\rho(T,\{>,\geq\}1-\gamma).\rho(\mathbf{d},\{\geq,>\}\gamma)$ |
| $\forall T.\mathbf{d}$ | $\triangleleft\gamma$ | $\exists\rho(T,\triangleleft^{-}1-\gamma).\rho(\mathbf{d},\triangleleft\gamma)$ |
| $\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$ | $\bowtie\gamma$ | $\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq m\}$ |
| $\geq m\ S.C$ | $\triangleright\gamma$ | $\geq m\ \rho(S,\triangleright\gamma).\rho(C,\triangleright\gamma)$ |
| $\geq m\ S.C$ | $\triangleleft\gamma$ | $\leq m{-}1\ \rho(S,\neg\triangleleft\gamma).\rho(C,\neg\triangleleft\gamma)$ |
| $\leq n\ S.C$ | $\{\geq,>\}\ \gamma$ | $\leq n\ \rho(S,\{>,\geq\}\ 1-\gamma).\rho(C,\{>,\geq\}\ 1-\gamma)$ |
| $\leq n\ S.C$ | $\triangleleft\gamma$ | $\geq n{+}1\ \rho(S,\triangleleft^{-}\ 1-\gamma).\rho(C,\triangleleft^{-}\ 1-\gamma)$ |
| $\geq m\ T.\mathbf{d}$ | $\triangleright\gamma$ | $\geq m\ \rho(T,\triangleright\gamma).\rho(\mathbf{d},\triangleright\gamma)$ |
| $\geq m\ T.\mathbf{d}$ | $\triangleleft\gamma$ | $\leq m{-}1\ \rho(T,\neg\triangleleft\gamma).\rho(\mathbf{d},\neg\triangleleft\gamma)$ |
| $\leq n\ T.\mathbf{d}$ | $\{\geq,>\}\ \gamma$ | $\leq n\ \rho(T,\{>,\geq\}\ 1-\gamma).\rho(\mathbf{d},\{>,\geq\}\ 1-\gamma)$ |
| $\leq n\ T.\mathbf{d}$ | $\triangleleft\gamma$ | $\geq n{+}1\ \rho(T,\triangleleft^{-}\ 1-\gamma).\rho(\mathbf{d},\triangleleft^{-}\ 1-\gamma)$ |
| $\exists S.Self$ | $\triangleright\gamma$ | $\exists\rho(S,\triangleright\gamma).Self$ |
| $\exists S.Self$ | $\triangleleft\gamma$ | $\neg\exists\rho(S,\neg\triangleleft\gamma).Self$ |
| $R_A$ | $\triangleright\gamma$ | $R_{A\triangleright\gamma}$ |
| $R_A$ | $\triangleleft\gamma$ | $\neg R_{A\neg\triangleleft\gamma}$ |
| $T$ | $\triangleright\gamma$ | $T_{\triangleright\gamma}$ |
| $T$ | $\triangleleft\gamma$ | $\neg T_{\neg\triangleleft\gamma}$ |
| $R^{-}$ | $\bowtie\gamma$ | $\rho(R,\bowtie\gamma)^{-}$ |
| $U$ | $\triangleright\gamma$ | $U$ |
| $U$ | $\triangleleft\gamma$ | $\neg U$ |
| $\mathbf{d}$ | $\geq\alpha$ | $real[a+\alpha(b-a), d-\alpha(d-c)]$ |
| $\mathbf{d}$ | $>\beta$ | $real[a+\beta(b-a)+\epsilon, d-\beta(d-c)-\epsilon]$ |
| $\mathbf{d}$ | $\leq\beta$ | $union\text{–}real[k_1, a+\beta(b-a), d-\beta(d-c), k_2]$ |
| $\mathbf{d}$ | $<\alpha$ | $union\text{–}real[k_1, a+\alpha(b-a)-\epsilon, d-\alpha(d-c)+\epsilon, k_2]$ |

Table 5. Reduction of the axioms.

| | |
|---|---|
| $\kappa(\langle a : C \bowtie \gamma \rangle)$ | $\{a : \rho(C, \bowtie \gamma)\}$ |
| $\kappa(\langle (a,b) : R \bowtie \gamma \rangle)$ | $\{(a,b) : \rho(R, \bowtie \gamma)\}$ |
| $\kappa(\langle (a,v) : T \bowtie \gamma \rangle)$ | $\{(a,v) : \rho(T, \bowtie \gamma)\}$ |
| $\kappa(\langle a \neq b \rangle)$ | $\{a \neq b\}$ |
| $\kappa(\langle a = b \rangle)$ | $\{a = b\}$ |
| $\kappa(C \sqsubseteq D \geq \alpha)$ | $\bigcup_{\gamma \in N^{fK} \setminus \{0\} \mid \gamma \leq \alpha} \{\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)\}$ $\bigcup_{\gamma \in N^{fK} \mid \gamma < \alpha} \{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\}$ |
| $\kappa(C \sqsubseteq D > \beta)$ | $\kappa(C \sqsubseteq D \geq \beta) \cup \{\rho(C, > \beta) \sqsubseteq \rho(D, > \beta)\}$ |
| $\kappa(\langle R_1 \dots R_n \sqsubseteq R \geq \alpha \rangle)$ | $\bigcup_{\gamma \in N^{fK} \setminus \{0\} \mid \gamma \leq \alpha} \{\rho(R_1, \geq \gamma) \dots \rho(R_n, \geq \gamma) \sqsubseteq \rho(R, \geq \gamma)\}$ $\bigcup_{\gamma \in N^{fK} \mid \gamma < \alpha} \{\rho(R_1, > \gamma) \dots \rho(R_n, > \gamma) \sqsubseteq \rho(R, > \gamma)\}$ |
| $\kappa(\langle R_1 \dots R_n \sqsubseteq R > \beta \rangle)$ | $\kappa(\langle R_1 \dots R_n \sqsubseteq R \geq \beta \rangle) \cup \{\rho(R_1, > \beta) \dots \rho(R_n, > \beta) \sqsubseteq \rho(R, > \beta)\}$ |
| $\kappa(dis(S_1, S_2))$ | $\{dis(\rho(S_1, > 0), \rho(S_2, > 0))\}$ |
| $\kappa(ref(R))$ | $\{ref(\rho(R, \geq 1))\}$ |
| $\kappa(asy(S))$ | $\{asy(\rho(S, > 0)\}$ |

**Example 2.** Let us consider some cases of the reduction (see the Appendix for further details).

- Consider an assertion $\langle a : \forall R.C \geq \alpha \rangle$. If it is satisfied, there exists a fuzzy interpretation $\mathcal{I}$ such that $\inf_{y \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, y), C^{\mathcal{I}}(y)\} \geq \alpha$. For an arbitrary $y$, $R^{\mathcal{I}}(a^{\mathcal{I}}, y) \leq 1 - \alpha$ or $C^{\mathcal{I}}(y) \geq \alpha$ must hold. Hence, if $R^{\mathcal{I}}(a^{\mathcal{I}}, y) \leq 1 - \alpha$ is not satisfied (i.e., $R^{\mathcal{I}}(a^{\mathcal{I}}, y) > 1 - \alpha$), then we deduce that $C^{\mathcal{I}}(y) \geq \alpha$, which is the semantics of the crisp assertion $a : \forall \rho(R, > 1 - \alpha).\rho(C, \geq \alpha)$.
- Consider $\langle a : (\geq m\ S.C) \leq \beta \rangle$. If it is satisfied, it follows that $\sup_{y_1, \dots, y_m \in \Delta^{\mathcal{I}}} (\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, y_i) \otimes C^{\mathcal{I}}(y_i)\}) \leq \beta$, so there cannot exist $m$ different individuals $y_i$ with $(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, y_i) \otimes C^{\mathcal{I}}(y_i)\}) > \beta$, and it follows that the crisp assertion $a : \leq m{-}1\ \rho(S, > \beta).\rho(C, > \beta)$ is satisfied.
- Consider $\langle C \sqsubseteq D \geq \alpha \rangle$. If it is satisfied, $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$. As this is true for the infimum, an arbitrary $x \in \Delta^{\mathcal{I}}$ must satisfy $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$. From the semantics of the Gödel implication, this is true if $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ or $D^{\mathcal{I}}(x) \geq \alpha$. Hence, for each $\gamma \in N^{fK} \setminus \{0\}$ such that $\gamma \leq \alpha$, $C^{\mathcal{I}}(x) \geq \gamma$ implies $D^{\mathcal{I}}(x) \geq \gamma$ (which is expressed as $\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)$) and for each $\gamma \in N^{fK} \mid \gamma < \alpha$, $C^{\mathcal{I}}(x) > \alpha$ implies $D^{\mathcal{I}}(x) > \alpha$ $(\rho(C, > \alpha) \sqsubseteq \rho(D, > \alpha))$. □

Summing up, a fKB $fK = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced into a KB $\mathcal{K}(fK) = \langle \kappa(\mathcal{A}), T(N^{fK}) \cup \kappa(\mathcal{T}), R_a(N^{fK}) \cup R_c(N^{fK}) \cup \kappa(\mathcal{R}) \rangle$. We highlight that the reduction preserves simplicity of the roles and regularity of the RIAs. Now we will illustrate the procedure with an example.

**Example 3.** Consider a travel agency, offering accommodation to the audience of some language courses. We suppose a fuzzy KB $fKB$ representing the following knowledge:

- $h_1$ is a hotel located at a German speaking country, being this axiom true with at least degree 0.75,
- the price of $h_1$ can be defined using a trapezoidal function $trap_{0,150}(x; 0, 50, 100, 150)$ with at least degree 0.5,
- some individual is close to another if and only if the latter is close to the former.
- $h_2$ is not a hotel.
- $h_1$ and $h_2$ are close with a degree not greater than 0.5.

Then, $fKB = \{\mathcal{A}, \emptyset, \mathcal{R}\}$ with $\mathcal{A} = \{\langle h_1 : Hotel \sqcap \exists isIn.\{(germany, 1), (austria, 1), (switzerland, 0.67)\} \geq 0.75\rangle, \langle h_1 \exists hasPrice.trap_{0,150}(x; 0, 50, 100, 150) \geq 0.5\rangle, \langle h_2 : \neg Hotel \geq 1\rangle, \langle (h_1, h_2) : isCloseTo \leq 0.5\rangle\}$ and $\mathcal{R} = \{sym(isCloseTo)\}$.

Firstly, $\langle sym(isCloseTo)\rangle$ is replaced by the fuzzy RIA $\langle isCloseTo \sqsubseteq isCloseTo^{-} \geq 1\rangle$. Then, we compute the number of truth values which have to be considered: $X^{fK} = \{0, 0.5, 1, 0.75\}$, so $N^{fK} = \{0, 0.25, 0.5, 0.75, 1\}$.

Next, we create some new elements and some axioms preserving their semantics. The new axioms due to the new concepts are $T(N^{fK}) = \{Hotel_{\geq 1} \sqsubseteq Hotel_{>0.75}$, $Hotel_{>0.75} \sqsubseteq Hotel_{\geq 0.75}$, $Hotel_{\geq 0.75} \sqsubseteq Hotel_{>0.5}$, $Hotel_{>0.5} \sqsubseteq Hotel_{\geq 0.5}$, $Hotel_{\geq 0.5} \sqsubseteq Hotel_{>0.25}$, $Hotel_{>0.25} \sqsubseteq Hotel_{\geq 0.25}$ and $Hotel_{\geq 0.25} \sqsubseteq Hotel_{>0}\}$.

The new axioms generated for the concrete role *hasPrice* are: $R_c(N^{fK}) = \{hasPrice_{\geq 1} \sqsubseteq hasPrice_{>0.75}$, $hasPrice_{>0.75} \sqsubseteq hasPrice_{\geq 0.75}$, $hasPrice_{\geq 0.75} \sqsubseteq hasPrice_{>0.5}$, $hasPrice_{>0.5} \sqsubseteq hasPrice_{\geq 0.5}$, $hasPrice_{\geq 0.5} \sqsubseteq hasPrice_{>0.25}$, $hasPrice_{>0.25} \sqsubseteq hasPrice_{\geq 0.25}$ and $hasPrice_{\geq 0.25} \sqsubseteq hasPrice_{>0}\}$.

Similarly, $R_a(N^{fK}) = \{isIn_{\geq 1} \sqsubseteq isIn_{>0.75}$, $isIn_{>0.75} \sqsubseteq isIn_{\geq 0.75}$, $isIn_{\geq 0.75} \sqsubseteq isIn_{>0.5}$, $isIn_{>0.5} \sqsubseteq isIn_{\geq 0.5}$, $isIn_{\geq 0.5} \sqsubseteq isIn_{>0.25}$, $isIn_{>0.25} \sqsubseteq isIn_{\geq 0.25}$, $isIn_{\geq 0.25} \sqsubseteq isIn_{>0}, isCloseTo_{\geq 1} \sqsubseteq isCloseTo_{>0.75}, isCloseTo_{>0.75} \sqsubseteq isCloseTo_{\geq 0.75}$, $isCloseTo_{\geq 0.75} \sqsubseteq isCloseTo_{>0.5}, isCloseTo_{>0.5} \sqsubseteq isCloseTo_{\geq 0.5}, isCloseTo_{\geq 0.5} \sqsubseteq isCloseTo_{>0.25}, isCloseTo_{>0.25} \sqsubseteq isCloseTo_{\geq 0.25}$ and $isCloseTo_{\geq 0.25} \sqsubseteq isCloseTo_{>0}\}$.

Finally, we map the four axioms in the ABox, and the axiom in the RBox:

- $\kappa(\langle h_1 : Hotel \sqcap \exists isIn.\{(germany, 1), (austria, 1), (switzerland, 0.67)\} \geq 0.75\rangle) = h_1 : Hotel_{\geq 0.75} \sqcap \exists isIn_{\geq 0.75}.\{germany, austria\}$.
- $\kappa(\langle h_1 : \exists hasPrice.trap_{0,150}(x; 0, 50, 100, 150) \geq 0.5\rangle) = h_1 : \exists hasPrice_{\geq 0.5}.real[25, 125]$. We also include the following definition of $real[25, 125]$:

```
<owl:DataRange rdf:about="#real[25,125]">
  <owl11:onDataRange rdf:resource="&xsd;double"/>
  <owl11:minInclusive rdf:datatype="&xsd;double">25</owl11:minInclusive>
  <owl11:maxInclusive rdf:datatype="&xsd;double">125</owl11:maxInclusive>
</owl:DataRange>
```

- $\kappa(\langle h_2 : \neg Hotel \geq 1\rangle) = h_2 : \rho(\neg Hotel, \geq 1) = h_2 : \neg Hotel_{>0}$.

- $\kappa(\langle (h_1, h_2) : isCloseTo \leq 0.5 \rangle) = (h_1, h_2) : \neg isCloseTo_{>0.5}$.
- $\kappa(\langle isCloseTo \sqsubseteq isCloseTo^- \geq 1 \rangle) = \{isCloseTo_{>0} \sqsubseteq isCloseTo^-_{>0}, isCloseTo_{\geq 0.25} \sqsubseteq isCloseTo^-_{\geq 0.25}, isCloseTo_{>0.25} \sqsubseteq isCloseTo^-_{>0.25}, isCloseTo_{\geq 0.5} \sqsubseteq isCloseTo^-_{\geq 0.5}, isCloseTo_{>0.5} \sqsubseteq isCloseTo^-_{>0.5}, isCloseTo_{\geq 0.75} \sqsubseteq isCloseTo^-_{\geq 0.75}, isCloseTo_{>0.75} \sqsubseteq isCloseTo^-_{>0.75}, isCloseTo_{\geq 1} \sqsubseteq isCloseTo^-_{\geq 1}\}$. □

### 4.3. *Optimizing GCI reductions*

In some particular (but important cases), the reduction of a fuzzy GCI can be optimized:

- $\langle C \sqsubseteq \top \triangleright \gamma \rangle$ and $\langle \bot \sqsubseteq D \triangleright \gamma \rangle$ are tautologies, so their reductions are unnecessary in the resulting KB.
- $\kappa(\langle \top \sqsubseteq D \triangleright \gamma \rangle) = \{\top \sqsubseteq \rho(D, \triangleright\gamma)\}$. Note that this kind of axiom appears in role range axioms i.e., $C$ is the range of $R$ iff $\top \sqsubseteq \forall R.C$ holds with degree 1.
- $\kappa(\langle C \sqsubseteq \bot \triangleright \gamma \rangle) = \{\rho(C, > 0) \sqsubseteq \bot\}$. This appears when two concepts are disjoint i.e., $C$ and $D$ are disjoint iff $C \sqcap D \sqsubseteq \bot$ holds with degree 1.
- The reduction of GCIs including fuzzy nominals can be optimized by taking advantage of the following observation. Suppose that the resulting TBox contains three GCIs $A \sqsubseteq B$, $A \sqsubseteq C$ and $B \sqsubseteq C$. Then, $A \sqsubseteq C$ is unnecessary because it can be deduced from the other two axioms.

**Example 4.** The reduction of the fuzzy GCI $\langle C \sqsubseteq \{1/o_1, 0.5/o_2\} \rangle$ is $\{C_{>0} \sqsubseteq \{o_1, o_2\}, C_{\geq 0.5} \sqsubseteq \{o_1, o_2\}, C_{>0.5} \sqsubseteq \{o_1\}, C_{\geq 1} \sqsubseteq \{o_1\}\}$. However, it can be optimized to $\{C_{>0} \sqsubseteq \{o_1, o_2\}, C_{>0.5} \sqsubseteq \{o_1\}\}$. □

### 4.4. *Properties of the reduction*

The following theorem shows that the reductions preserves reasoning.

**Theorem 1.** *A $f_{KD}\mathcal{SROIQ}(\mathbf{D})$ fKB $fK$ is satisfiable iff $\mathcal{K}(fK)$ is satisfiable.*

**Proof.** Proof can be found in an Appendix. □

An interesting property of the procedure is that, under certain conditions given by Theorem 2, the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included. From an implementation point of view, this property allows to compute $\mathcal{K}(fK)$ off-line and update it incrementally. We note that the reduction of the new axioms may include the definitions of new customized datatypes in case concrete concepts appear in them.

**Theorem 2.** *Let $fK$ be a $f_{KD}\mathcal{SROIQ}(\mathbf{D})$ fuzzy knowledge base involving a set of fuzzy atomic roles $\mathbf{A}$, a set of a set of atomic roles $\mathbf{R_a}$ and a set of concrete roles $\mathbf{R_c}$, let $N^{fK}$ be the set of relevant degrees to be considered and let $\tau$ be a $f_{KD}\mathcal{SROIQ}(\mathbf{D})$ axiom such that:*

(1) *for every atomic concept $A$ which appears in $\tau$, $A \in \mathbf{A}$,*
(2) *for every atomic role $R_A$ which appears in $\tau$, $R_A \in \mathbf{R_a}$,*
(3) *for every concrete role $T$ which appears in $\tau$, $T \in \mathbf{R_c}$,*
(4) *if $\gamma$ appears in $\tau$, then $\gamma \in N^{fK}$.*

*Then, $\mathcal{K}(fK \cup \tau) = \mathcal{K}(fK) \cup \mathcal{K}(\tau)$.*

**Proof.** Trivial from the following observations:

- Every axiom is reduced to a combination of new crisp elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fKB.
- $\tau$ does not introduce atomic concepts, atomic abstract roles, concrete roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others. □

The theorem assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often, which are reasonable assumptions because ontologies do not usually change once that their developement has finished, and because it has been shown that the set of the degrees which must be considered for any reasoning task is $N^{fK}$.[12] Consequently, even in case of an entailment test, it makes sense to use a degree in $N^{fK}$. Regarding the computation of the greatest lower bound, we recall that U. Straccia has shown that, in the worst case, it requires to compute $log|N^{fK}|$ satisfiability tests,[25] which is another argument to fix the set of allowed degrees.

### 4.5. *Complexity*

$|\mathcal{K}(fK)|$ is $O(|fK|^2)$ i.e., the resulting knowledge base is quadratic. The ABox is actually linear while the TBox and the RBox are both quadratic:

- $|N^{fK}|$ is linearly bounded by $|\mathcal{A}| + |\mathcal{T}| + |\mathcal{R}|$,
- $|\kappa(\mathcal{A})| = |\mathcal{A}|$,
- $|T(N^{fK})| = 2 \cdot (|N^{fK}| - 1) \cdot |\mathbf{A}| - 1$,
- $|\kappa(\mathcal{T})| \leq 2 \cdot (|N^{fK}| - 1) \cdot |\mathcal{T}|$,
- $|R_a(N^{fK})| = 2 \cdot (|N^{fK}| - 1) \cdot |\mathbf{R}| - 1$,
- $|R_c(N^{fK})| = 2 \cdot (|N^{fK}| - 1) \cdot |\mathbf{T}| - 1$,
- $|\kappa(\mathcal{R})| \leq 2 \cdot (|N^{fK}| - 1) \cdot |\mathcal{R}|$.

The resulting KB is quadratic because it depends on the number of relevant degrees $|N^{fK}|$. An immediate solution to obtain a KB which is linear in complexity is to fix the number of degrees which can appear in the knowledge base. From a practical point of view, in most of the applications it is sufficient to consider a small number of degrees, e.g. $\{0, 0.25, 0.5, 0.75, 1\}$, i.e., $\alpha \in \{0.25, 0.5, 0.75, 1\}$

and $\beta \in \{0, 0.25, 0.5, 0.75\}$. Then, the size of the resulting KB is linear since now $|T(N^{fK})| = 7 \cdot |\mathbf{A}|$, $|\kappa(\mathcal{T})| \leq 8 \cdot |\mathcal{T}|$, $|R_a(N^{fK})| = 7 \cdot |\mathbf{R}|$, $|R_c(N^{fK})| = 7 \cdot |\mathbf{T}|$, $|\kappa(\mathcal{R})| \leq 8 \cdot |\mathcal{R}|$.

### 4.6. *Allowing crisp concepts and roles*

It is easy to see that the complexity of the crisp representation is caused by fuzzy atomic concepts and roles. Fortunately, in real applications not all concepts and roles will be fuzzy. Therefore, an interesting optimization is allowing to specify that an atomic concept (resp. an atomic abstract role, a concrete role) is crisp. For instance, suppose that $A$ is a fuzzy atomic concept. Then, we need $|N^{fK}| - 1$ concepts of the form $A_{\geq\alpha}$ and another $|N^{fK}| - 1$ concepts of the form $A_{>\beta}$ to represent it, as well as $2(|N^{fK}| - 1)$ axioms to preserve their semantics. On the other hand, if $A$ is declared to be crisp, we just need one crisp concept $A_{crisp}$ to represent it and no new axioms. The case for atomic abstract roles $R_a$ (resp. concrete roles $T$, since complex concrete roles are not allowed) is similar, only one crisp element $R_{acrisp}$ (resp. $T_{crisp}$) is needed.

Handling this crisp elements is very easy, because we only need to consider the following extension of $\rho$ for those elements asserted to be interpreted as crisp:

| $x$ | $y$ | $\rho(x,y)$ |
|---|---|---|
| $A$ | $\rhd\gamma$ | $A_{crisp}$ |
| $A$ | $\lhd\gamma$ | $\neg A_{crisp}$ |
| $R_A$ | $\rhd\gamma$ | $R_{acrisp}$ |
| $R_A$ | $\lhd\gamma$ | $\neg R_{acrisp}$ |
| $T$ | $\rhd\gamma$ | $T_{crisp}$ |
| $T$ | $\lhd\gamma$ | $\neg T_{crisp}$ |

### 4.7. *Optimizing the reasoning*

Our reduction is optimized to promote reusing in the conditions shown in Theorem 2. However, before a satisfiability test is performed, some axioms do not need to be considered. These axioms cannot be removed from the crisp KB, because they may be necessary when new axioms are added to it, but they are superfluous for computing the satisfiability of the crisp KB.

We say that that an atomic concept $A$ is *superfluous* for reasoning in $\mathcal{K}(fK) = \langle \kappa(\mathcal{A}), T(N^{fK}) \cup \kappa(\mathcal{T}), R_a(N^{fK}) \cup R_c(N^{fK}) \cup \kappa(\mathcal{R}) \rangle$ if it appears in $T(N^{fK})$ but it does not appear in the other parts of $\mathcal{K}(fK)$. The intuition here is that superfluous concepts cannot cause a contradiction. Hence, in order to reason with $\mathcal{K}(fK)$, we may replace $T(N^{fK})$ with $T'(N^{fK})$ such that it does not contain any superfluous concept. For each $1 \leq i \leq |N^{fK}| - 1, 2 \leq j \leq |N^{fK}| - 1$ and for each $A \in \mathbf{A}$, $T'(N^{fK})$ is the smallest terminology containing $A_{\geq\gamma_{i+1}} \sqsubseteq A_{>\gamma_i}$ and $A_{>\gamma_j} \sqsubseteq A_{\geq\gamma_j}$ such that $A_{\geq\gamma_{i+1}}, A_{>\gamma_i}, A_{>\gamma_j}, A_{\geq\gamma_j}$ are not superfluous. Note that if additional axioms are added to $fK$, $\mathcal{K}(fK)$ will be different and $A$ may stop being superfluous.

We can proceed similarly with roles. An atomic role $R_a$ is *superfluous* for reasoning in $\mathcal{K}(fK) = \langle \kappa(\mathcal{A}), T(N^{fK}) \cup \kappa(\mathcal{T}), R_a(N^{fK}) \cup R_c(N^{fK}) \cup \kappa(\mathcal{R})\rangle$ if it appears in $R(N^{fK})$ but it does not appear in the other parts of $\mathcal{K}(fK)$. Once again, this may no longer hold when new axioms are added to the original fKB. Before reasoning with $\mathcal{K}(fK)$, we may replace $R_a(N^{fK})$ with $R'_a(N^{fK})$, which is defined as the smallest terminology containing, for each $R_A \in \mathbf{R}$, the axioms $R_{A\geq\gamma_{i+1}} \sqsubseteq R_{A>\gamma_i}$ and $R_{A>\gamma_i} \sqsubseteq R_{A\geq\gamma_i}$, such that $R_{A\geq\gamma_{i+1}}, R_{A>\gamma_i}, R_{A>\gamma_i}, R_{A\geq\gamma_i}$ are not superfluous. We can proceed similarly with abstract roles $T$, and define $R'_c(N^{fK})$ from $R_c(N^{fK})$.

**Proposition 3.** $\mathcal{K}(fK) = \langle \kappa(\mathcal{A}), T(N^{fK}) \cup \kappa(\mathcal{T}), R_a(N^{fK}) \cup R_c(N^{fK}) \cup \kappa(\mathcal{R})\rangle$ *is satisfiable iff* $\mathcal{K}(fK) = \langle \kappa(\mathcal{A}), T'(N^{fK}) \cup \kappa(\mathcal{T}), R'_a(N^{fK}) \cup R'_c(N^{fK}) \cup \kappa(\mathcal{R})\rangle$ *is satisfiable.*

**Proof.** Trivial from the fact that superfluous elements cannot cause contradictions with elements not belonging to $T(N^{fK})$, $R_a(N^{fK})$ and $R_c(N^{fK})$. □

**Example 5.** Consider the KB $\mathcal{K}(fK)$ obtained in Example 3. The procedure has created atomic concepts $Hotel_{\geq 1}$, $Hotel_{>0.75}$, $Hotel_{\geq 0.75}$, $Hotel_{>0.5}$, $Hotel_{\geq 0.5}$, $Hotel_{>0.25}$, $Hotel_{\geq 0.25}$ and $Hotel_{>0}$. However, if we consider the KB without $T^{fK}$, it only contains $Hotel_{\geq 0.75}$ and $Hotel_{>0}$. Hence, the other concepts are superfluous, so we may replace $T(N^{fK})$ with $T'(N^{fK}) = \{Hotel_{\geq 0.75} \sqsubseteq Hotel_{>0}\}$. □

## 5. Related Work

There has been a relatively significant amount of work in extending ontologies[9] and DLs[10] with fuzzy set theory. In this section we will concentrate on the state of the art on fuzzy inclusions and the representation of fuzzy DLs using crisp DLs.

**Fuzzy GCIs and RIAs.** The most used semantics for fuzzy GCIs is based on Zadeh's set inclusion: $C \sqsubseteq D = \forall x \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$.[25] However, this approach forces concept inclusion to be a yes-no question, which we think that it is sometimes too restrictive. On the other hand, our approach allows for instance to express that the concept $Hotel$ subsumes the concept $Inn$ with degree 0.5. D. Dubois *et al.* proposed several levels of inclusion for a GCI, based on the core and support of fuzzy sets, $C \sqsubseteq D$: *(i)* $core(C) \subseteq support(D)$, *(ii)* $core(C) \subseteq core(D)$ or $support(C) \subseteq support(D)$, *(iii)* $core(C) \subseteq core(D)$ and $support(C) \subseteq support(D)$, and *(iv)* $support(C) \subseteq core(D)$.[29] These definitions also force the inclusion to be either true or false; moreover, since in general $support(C) \not\subseteq core(C)$, using the latter semantics for GCIs implies that a concept does not fully subsume itself.

U. Straccia proposed to use implication functions in the semantics of fuzzy GCIs, making concept subsumption hold to a certain degree in $[0, 1]$.[21] F. Bobillo *et al.* proposed the use of KD implication in the semantics of GCIs,[13] providing the first reasoning algorithm with fuzzy GCIs. Unfortunately, although the use of KD implication in the semantics of fuzzy GCIs solves some of these problems, it brings about two counter-intuitive effects. Firstly, in general a concept does not fully subsume itself given that $C \sqsubseteq C = \inf_{x \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(x), C^{\mathcal{I}}(x)\} \geq 0.5$. Secondly, crisp concept subsumption (holding to degree 1) forces some fuzzy concepts to be interpreted as crisp since $\langle C \sqsubseteq D \geq 1 \rangle = \inf_{x \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \geq 1$ which is true iff for each element of the domain $D^{\mathcal{I}}(x) = 1$ or $1 - C^{\mathcal{I}}(x) \geq 1 \Rightarrow C^{\mathcal{I}}(x) = 0$. This problems can be solved using an $R$-implication (moreover, for any $R$-implication, the semantics of $\langle C \sqsubseteq D \geq 1 \rangle$ is equivalent to consider Zadeh's set inclusion). Some recent works consider Łukasiewicz implication[30] and Goguen (or product) implication,[31] in contrast to this work where we consider Gödel implication and show that a crisp representation of the fuzzy GCIs is possible.

Finally, although U. Straccia proposed to use implication functions also in the semantics of fuzzy RIAs, making role subsumption hold to a certain degree,[21] this is the first work supporting reasoning with them.

**Crisp representations for fuzzy DLs.** The first effort in this direction is due to U. Straccia, who showed a reasoning preserving procedure for fuzzy $\mathcal{ALCH}$.[12] Originally, we used this work as a starting point, though we have augmented the expressivity of the logic by adding new features (fuzzy nominals, fuzzy GCIs, fuzzy RIAs) to the language and even improved the method by reducing the number of new concepts, roles and axioms, as explained in Sec. 4. A similar work from him considers fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice,[32] therefore supporting quantitative reasoning (by using the interval $[0, 1]$) and qualitative reasoning (by relying on a set `{false, likelyfalse, unknown, likelytrue, true}`). F. Bobillo *et al.* extended the former work of U. Straccia to $\mathcal{SHOIN}$ and allowed fuzzy GCIs, but with a semantics given by KD implication[13] — which is not always intuitive as already mentioned. G. Stoilos *et al.* extended this work and considered the reduction of an extension of fuzzy $\mathcal{SHOIN}$ with additional role axioms: general RIAs, reflexive, asymmetric and role disjointness axioms.[14] It is not a reduction of fuzzy $\mathcal{SROIQ}$ (not even $\mathcal{SROIN}$) because they do not show how to reduce the universal role, qualified cardinality restrictions, local reflexivity concepts in expressions of the form $\rho(\exists S.Self, \triangleleft\gamma)$ nor negative role assertions. Moreover, GCIs and RIAs are forced to be either true or false. F. Bobillo *et al.* extended this work providing a crisp representation of full $\mathcal{SROIQ}$.[15]

A different approach is due to Y. Li *et al.*, who propose a family of fuzzy description logics using $\alpha$-cuts as atomic concept and roles.[33] The approach is slightly different to ours because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce an $\mathcal{ALCQ}$ ABox[34] and an $\mathcal{ALCH}$ concept[35] to their crisp versions. Nevertheless, both of these works assume an empty TBox. Finally, D. Dubois *et al.* combine possibilistic and fuzzy logics in the context of description logics (more concretely, in $\mathcal{ALCIN}(\circ)$).[29] Interestingly, they also propose to represent every fuzzy set using two crisp sets (its support and its core) and comment the possibility of extending their work by using more crisp sets, in order to have a more refined representation. Although for some applications this representation may be enough, there is a loss of information that does not occur in our approach.

## 6. Conclusions and Future Work

This work has presented an alternative approach to achieve fuzzy ontologies, which allows to reuse current crisp ontology languages and reasoners, among other related resources. It supposes an important step towards the possibility of dealing with imprecise and vague knowledge, since it relies on proved languages and tools.

Our work presents several contributions. Firstly, we augmented the expressiveness of the logic by adding fuzzy nominals, fuzzy GCIs and fuzzy RIAs with a semantics based on Gödel implication. This is the first work supporting reasoning with fuzzy RIAs. Secondly, we have shown how to reduce our fuzzy extension of full $\mathcal{SROIQ}(\mathbf{D})$ into crisp $\mathcal{SROIQ}(\mathbf{D})$. In particular, this is the first crisp representation for fuzzy concrete predicates. Interestingly, we have shown that, under some reasonable conditions, the reduction of a fKB can be reused when additional axioms are added to it. These requirements are to use the ontology vocabulary and to restrict the set of possible degrees of truth, since in practical applications it is usual to work with a small number of them. Restricting the degrees of truth turns also to be essential in order to compute the greatest lower bound and to make the resulting crisp KB be linear in size. Finally, we have optimized the reduction with respect to previous works, reducing the size of the resulting KB, optimizing some important cases of GCIs and removing some superfluous concepts before applying crisp reasoning. We have also discussed how to use crisp concepts and roles.

The main direction for future work is to perform an empirical evaluation of an implementation of this reduction with other fuzzy DL reasoners, although they support different languages and features and, as far as we know, there does not exist any significant fuzzy knowledge base. We also plan to augment the expressiveness of the logic by considering additional fuzzy operators.

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## Appendix A. Proof for Theorem 1

We will show the proof for the only-if direction. From $fK$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ with respect to a fuzzy concrete domain $\mathbf{D} = \langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Phi_{\mathbf{D}}$ only contains the fuzzy predicate $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$, satisfying every axiom in $fK$. Now, it is possible to build a (crisp) interpretation $\mathcal{I}_{\mathcal{C}} = \{\Delta^{\mathcal{I}_{\mathcal{C}}}, \cdot^{\mathcal{I}_{\mathcal{C}}}\}$ with respect to a crisp concrete domain $\mathbf{D_C} = \langle \Delta_{\mathbf{D_C}}, \Phi_{\mathbf{D_C}} \rangle$ as:

- $\Delta^{\mathcal{I}_{\mathcal{C}}} = \Delta^{\mathcal{I}}$.
- $\Delta_{\mathbf{D_C}} = \Delta_{\mathbf{D}}$.
- $x^{\mathcal{I}_{\mathcal{C}}} = x^{\mathcal{I}}$, for all $x \in \Delta^{\mathcal{I}}$.
- $v_{\mathbf{D_C}} = v_{\mathbf{D}}$, for all $v \in \Delta_{\mathbf{D}}$.
- $A_{\geq \alpha}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq \alpha\}$, for each $A \in fK$ and $\alpha \in N^{fK} \setminus \{0\}$.
- $A_{> \beta}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) > \beta\}$, for each $A \in fK$, $\beta \in N^{fK} \setminus \{1\}$.
- $R_{A \geq \alpha}^{\mathcal{I}_{\mathcal{C}}} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) \geq \alpha\}$, for each $R_A \in fK$, $\alpha \in N^{fK} \setminus \{0\}$.
- $R_{A > \beta}^{\mathcal{I}_{\mathcal{C}}} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) > \beta\}$, for each $R_A \in fK$, $\beta \in N^{fK} \setminus \{1\}$.
- $T_{\geq \alpha}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) \geq \alpha\}$, for each $T \in fK$, $\alpha \in N^{fK} \setminus \{0\}$.
- $T_{> \beta}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) > \beta\}$, for each $T \in fK$, $\beta \in N^{fK} \setminus \{1\}$.
- $\Phi_{\mathbf{D_C}}$ will contain some predicates of the form $real[a, b]$, $real(a, b)$ and $union\text{–}real[k_1, a, b, k_2]$, with $a, b, k_1, k_2 \in \mathbb{R}$.

Now, we will show that $\mathcal{I}_{\mathcal{C}}$ satisfies every axiom in $\mathcal{K}(fK)$. For every axiom $\tau \in fK$, there are several cases:

(1) $\tau$ is an equality assertion. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_{\mathcal{C}}} \neq b^{\mathcal{I}_{\mathcal{C}}}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a \neq b \rangle \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a \neq b \rangle)$. The case of equality assertions is similar.

(2) $\tau$ is a role assertion. Assume that $\mathcal{I} \models \langle (a, b) : R \bowtie \gamma \rangle$. We show, by induction on the structure of roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R \bowtie \gamma \rangle)$.

- *atomic role.* Assume that $\mathcal{I} \models \langle (a, b) : R_A \rhd \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) : R_{A \rhd \gamma}^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R_A, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(R_A, \rhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R_A \rhd \gamma \rangle)$.
  Now assume that $\mathcal{I} \models \langle (a, b) : R_A \lhd \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \lhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\neg R_{\neg \rhd \gamma})^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R_A, \lhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(R_A, \lhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R_A \lhd \gamma \rangle)$.
- *inverse role.* Assume that $\mathcal{I} \models \langle (a, b) : R^- \bowtie \gamma \rangle$. Then, $R^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}_{\mathcal{C}}}, a^{\mathcal{I}_{\mathcal{C}}}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R, \bowtie \gamma)^{\mathcal{I}_{\mathcal{C}}})^- \equiv \mathcal{I}_{\mathcal{C}} \models (a, b) \in \rho(R, \bowtie \gamma)^- \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a, b) : R^- \bowtie \gamma \rangle)$.
- *concrete roles.* This case is similar to the case of atomic roles.
- *universal role.* Assume that $\langle \mathcal{I} \models (a, b) : U \rhd \gamma \rangle$. Then, $U^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1 \geq \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in \Delta^{\mathcal{I}_{\mathcal{C}}} \times \Delta^{\mathcal{I}_{\mathcal{C}}}$ and consequently $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in U^{\mathcal{I}_{\mathcal{C}}} \equiv (a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(U, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models (a, b) : \rho(U, \rhd\gamma)$. The case $\mathcal{I} \models (a, b) : U \lhd \gamma \rangle$ is similar.

(3) $\tau$ is a concept assertion. Assume that $\mathcal{I} \models \langle a : C \bowtie \gamma \rangle$. We show, by induction on the structure of concepts and roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \bowtie \gamma \rangle)$.

- *atomic concept.* Assume that $\mathcal{I} \models \langle a : A \rhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_{\mathcal{C}}} : A^{\mathcal{I}_{\mathcal{C}}}_{\rhd \gamma}$. Consequently, $a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(A, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \rhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \rhd \gamma \rangle)$. Now assume that $\mathcal{I} \models \langle a : A \lhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_{\mathcal{C}}} \notin A^{\mathcal{I}_{\mathcal{C}}}_{\neg \rhd \gamma} \equiv a^{\mathcal{I}_{\mathcal{C}}} \in \neg A^{\mathcal{I}_{\mathcal{C}}}_{\neg \rhd \gamma} \equiv a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(A, \lhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \lhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \lhd \gamma \rangle)$.
- *top concept.* Assume that $\mathcal{I} \models \langle a : \top \rhd \gamma \rangle$. Then, $\top^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_{\mathcal{C}}} \in \Delta^{\mathcal{I}_{\mathcal{C}}} = \top$. Consequently, $a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(\top, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a : \rho(\top, \rhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \top \rhd \gamma \rangle)$. The case $\mathcal{I} \models \langle a : \top \lhd \gamma \rangle$ is not possible. If $\mathcal{I} \models \langle a : \top \leq \beta \rangle$ we have that $1 \leq \beta$, which is contradiction with the restriction $\beta \in [0, 1)$. If $\mathcal{I} \models \langle a : \top < \alpha \rangle$ we have that $1 < \alpha$, which is contradiction with the restriction $\alpha \in (0, 1]$.
- *bottom concept.* This case is similar to the previous one.
- *concept negation.* Assume that $\mathcal{I} \models \langle a : \neg C \bowtie \gamma \rangle$. Then, $1 - C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie^{-} 1 - \gamma$. By induction hypothesis, $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(C, \bowtie^{-} 1 - \gamma)^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a \in \rho(C, \bowtie^{-} 1 - \gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \neg C \bowtie \gamma \rangle)$.
- *concept conjunction.* Assume that $\mathcal{I} \models \langle a : C \sqcap D \rhd \gamma \rangle$. Then, $\min\{C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})\} \rhd \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$ and $D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(C, \rhd\gamma)^{\mathcal{I}_{\mathcal{C}}}$ and $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(D, \rhd\gamma)^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(C, \rhd\gamma) \sqcap \rho(D, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(C \sqcap D, \rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a : \rho(C \sqcap D, \rhd\gamma) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcap D \rhd \gamma \rangle)$.
  In the case $\mathcal{I} \models \langle a : C \sqcap D \lhd \gamma \rangle$, it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$ or $D^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(C, \lhd\gamma)^{\mathcal{I}_{\mathcal{C}}}$ or $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(D, \lhd\gamma)^{\mathcal{I}_{\mathcal{C}}}$. In this case, we end up with $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcap D \lhd \gamma \rangle)$.
- *concept disjunction.* It can be easily obtained using De Morgan laws.
- *universal quantification.* Assume that $\mathcal{I} \models \langle a : \forall R.C \geq \alpha \rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b), C^{\mathcal{I}}(b)\} \geq \alpha$. Since this is true for the infimum, an arbitrary individual $b \in \Delta^{\mathcal{I}}$ must satisfy $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leq 1 - \alpha$ or $C^{\mathcal{I}}(b) \geq \alpha$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}}, b) \in \rho(R, \leq 1 - \alpha)^{\mathcal{I}_{\mathcal{C}}}$ or $b \in \rho(C, \geq \alpha)^{\mathcal{I}_{\mathcal{C}}}$ for an arbitrary individual $b \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, which is equivalent to $a^{\mathcal{I}_{\mathcal{C}}} \in (\forall \rho(R, > 1 - \alpha).\rho(C, \geq \alpha))^{\mathcal{I}_{\mathcal{C}}} \equiv a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(\forall R.C \geq \alpha))^{\mathcal{I}_{\mathcal{C}}} \equiv \mathcal{I}_{\mathcal{C}} \models a : (\rho(\forall R.C \geq \alpha) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \forall R.C, \geq \alpha \rangle)$. The case $> \beta$ is quite straightforward.
  Now, assume that $\mathcal{I} \models \langle a : \forall R.C \leq \beta \rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b), C^{\mathcal{I}}(b)\} \leq \beta$. P. Hájek showed for Łukasiewicz logic (which is a more general case of Zadeh logic), that if there is a model, then there is also a witnessed model[36] so we can assume that, if this is true for the infimum, there exists an individual $b$ satisfying $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq 1 - \beta$ and $C^{\mathcal{I}}(b) \leq \beta$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}}, b) \in (\rho(R, \geq 1 - \beta))^{\mathcal{I}_{\mathcal{C}}}$ and $b \in (\rho(C, \leq \beta))^{\mathcal{I}_{\mathcal{C}}}$ for some individual $b \in \Delta^{\mathcal{I}_{\mathcal{C}}}$. In this case, we end up with $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \forall R.C, \leq \beta \rangle)$. The case $< \alpha$ is quite straightforward.
- *existential quantification.* Use inter-definability of quantifiers.

- *fuzzy nominals.* Assume that $\mathcal{I} \models \langle a : \{\alpha_1/o_1, \ldots, \alpha_n/o_n\} \rhd \gamma\rangle$. Let $o_{i1}, \ldots, o_{ik}$ be such that $\alpha_{ij} \rhd \gamma$. Then, $\sup\{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \rhd \gamma$, with $a^{\mathcal{I}} \in \{o_{i_1}, \ldots, o_{i_k}\}^{\mathcal{I}}$. By construction of $\mathcal{I_C}$, it holds that $a^{\mathcal{I_C}} \in \{o_{i_1}, \ldots, o_{i_k}\}^{\mathcal{I_C}} \equiv a^{\mathcal{I_C}} \in \rho(\{\alpha_1/o_1, \ldots, \alpha_n/o_n\}^{\mathcal{I_C}}, \rhd\gamma) \equiv \mathcal{I_C} \models a : \rho(\{\alpha_1/o_1, \ldots, \alpha_n/o_n\}, \rhd\gamma))) \equiv \mathcal{I_C} \models \kappa(\langle a : \{\alpha_1/o_1, \ldots, \alpha_n/o_n\} \rhd \gamma\rangle)$. The case $\lhd\gamma$ is quite straightforward.
- *at-least qualified number restriction.* Assume that $\mathcal{I} \models \langle a : (\geq m\ S.C) \geq \alpha\rangle$. Then, $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes (\otimes_{j<k}\{b_j \neq b_k\})] \geq \alpha$. Note that $(\otimes_{j<k}\{b_j \neq b_k\})$ can be either 0 or 1. If it is 0, then we have that $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes 0] = 0 \geq \alpha$, which is not possible because by definition $\alpha \in (0, 1]$. Hence, $(\otimes_{j<k}\{b_j \neq b_k\}) = 1$ and consequently $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes 1] = \sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{m}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \geq \alpha$. This implies that there exist $m$ different $b_i \in^{\mathcal{I_C}}$ such that $\otimes_{i=1}^{m}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}$ and hence $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \geq \alpha$ and $C^{\mathcal{I}}(b_i) \geq \alpha$, for $1 \leq i \leq m$. By induction hypothesis, $(a^{\mathcal{I_C}}, b_i) \in (\rho(S, \geq \alpha))^{\mathcal{I_C}}$ and $b_i \in (\rho(C, \geq \alpha))^{\mathcal{I_C}}$, for $1 \leq i \leq m$. Consequently, $a^{\mathcal{I_C}} \in (\geq m\ \rho(S, \geq \alpha).\rho(C, \geq \alpha))^{\mathcal{I_C}} \equiv a^{\mathcal{I_C}} \in \rho(\geq m\ S.C, \geq \alpha)^{\mathcal{I_C}} \equiv \mathcal{I_C} \models a : \rho(\geq m\ S.C, \geq \alpha) \equiv \mathcal{I_C} \models \kappa(\langle a : (\geq m\ S.C) \geq \alpha\rangle)$. The case $> \beta$ is quite similar.
  Now assume that $\mathcal{I} \models \langle a : (\geq m\ S.C) \leq \beta\rangle$. In this case, it follows that $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} (\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I_C}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \leq \beta$. Consequently, there cannot exist $m$ different individuals $b_i$ with $(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I_C}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) > \beta$, so we end up with $\mathcal{I_C} \models \kappa(\langle a : (\geq m\ S.C) \leq \beta\rangle)$. The case $< \alpha$ is quite similar.
- *at-most qualified number restriction.* It can be easily obtained using interdefinability of qualified cardinality restrictions.
- *local reflexivity.* Assume that $\mathcal{I} \models \langle a : \exists S.Self \rhd \gamma\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $(a^{\mathcal{I_C}}, a^{\mathcal{I_C}}) \in \rho(S, \rhd\gamma)^{\mathcal{I_C}} \equiv \mathcal{I_C} \models (a, a) : \rho(S, \rhd\gamma) \equiv \mathcal{I_C} \models \kappa(\langle a : \exists S.Self \rhd \gamma\rangle)$. Now assume that $\mathcal{I} \models \langle a : \exists S.Self \lhd \gamma\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $(a^{\mathcal{I_C}}, a^{\mathcal{I_C}}) \in \rho(S, \lhd\gamma)^{\mathcal{I_C}}$. Hence, it follows that $(a^{\mathcal{I_C}}, a^{\mathcal{I_C}}) \notin (\rho(S, \neg \lhd \gamma))^{\mathcal{I_C}} \equiv (a^{\mathcal{I_C}}, a^{\mathcal{I_C}}) \in \neg(\rho(S, \neg \lhd \gamma))^{\mathcal{I_C}} \equiv a^{\mathcal{I_C}} \in (\rho(\exists S.Self, \lhd\gamma))^{\mathcal{I_C}} \equiv \mathcal{I_C} \models a : \rho(\exists S.Self, \lhd\gamma) \equiv \mathcal{I_C} \models \kappa(\langle a : \exists S.Self, \lhd\gamma\rangle)$.
- *concrete concept constructs.* Concrete concept constructs are similar to the abstract versions. The only difference is that some expressions of the form $v_i : d_{\mathbf{D}} \bowtie \gamma$ may appear, with $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$. Assume that $v_i : d_{\mathbf{D}} > \beta$. In order to guarantee that the trapezoidal function takes a value $x_{v_i}$ which is greater or equal than $\beta$, we have that $x_{v_i} > a + \beta(b-a)$ and $x_{v_i} < d - \beta(d-c)$, which is equivalent to say that $v_i \in real[a + \beta(b-a) + \epsilon, d - \beta(d-c) - \epsilon]$. The case $\geq \alpha$ is similar. Now, assume that $v_i : d_{\mathbf{D}} \leq \beta$. In this case we have that either $x_{v_i} \leq k_1, a + \beta(b-a)$ or $x_{v_i} \geq d - \beta(d-c), k_2]$, which is equivalent to say that $v_i \in union\text{–}real[k_1, a + \beta(b-a), d - \beta(d-c), k_2]$. The case $< \alpha$ is similar.

(4) $\tau$ is a fuzzy GCI. Assume that $\mathcal{I} \models \langle C \sqsubseteq D \geq \alpha \rangle$. Then, $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$. Hence, for an arbitrary individual $x \in \Delta^{\mathcal{I}}$ it follows that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$ and hence one of the following conditions holds *(i)* $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ (which makes the Gödel implication equal to $1 \geq \alpha$), or *(ii)* $D^{\mathcal{I}}(x) \geq \alpha$ (which makes the Gödel implication take a value $\geq \alpha$). Note that the former condition is equivalent to: $C^{\mathcal{I}}(x) \rhd \gamma$ implies $D^{\mathcal{I}}(x) \rhd \gamma$ for every $\gamma \in N^{fK}$. By induction hypothesis, it follows that $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(C, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}$ implies $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(D, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}$ or $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(D, \geq \alpha))^{\mathcal{I}_{\mathcal{C}}}$, for an arbitrary $x \in \Delta^{\mathcal{I}_{\mathcal{C}}}$. Consequently, it follows that $\mathcal{I}_{\mathcal{C}} \models \bigcup_{\gamma \in N^{fK} \setminus \{0\} \mid \gamma \leq \alpha} \{\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)\} \bigcup_{\gamma \in N^{fK} \mid \gamma < \alpha} \{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\} \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(\langle C \sqsubseteq D \geq \alpha \rangle)$. The case for $> \beta$ is quite similar.

(5) $\tau$ is a fuzzy RIA. Assume that $\mathcal{I} \models \langle R_1 \dots R_n \sqsubseteq R \rhd \gamma \rangle$. The case is similar to the previous one, with the difference that there appears a minimum i.e., $\min\{R_1^{\mathcal{I}}(y_1, y_2), \dots, R_n^{\mathcal{I}}(y_n, y_{n+1})\} \leq \{R^{\mathcal{I}}(y_1, y_{n+1})$. As a consequence, the left side of the crisp RIAs will contain $\rho(R_1, \rhd\gamma) \dots \rho(R_n, \rhd\gamma)$ in the left side, instead of $\rho(C, \rhd\gamma)$.

(6) $\tau$ is a role disjoint axiom. Assume that $\mathcal{I} \models dis(S_1, S_2)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = 0$ or $S_2^{\mathcal{I}}(x, y) = 0$. By induction hypothesis, $\forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, y) \in (\rho(S_1, \leq 0))^{\mathcal{I}_{\mathcal{C}}}$ or $(x, y) \in (\rho(S_2, \leq 0))^{\mathcal{I}_{\mathcal{C}}} \equiv \forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, y) \notin (\rho(S_1, > 0))^{\mathcal{I}_{\mathcal{C}}}$ or $(x, y) \notin (\rho(S_2, > 0))^{\mathcal{I}_{\mathcal{C}}} \equiv (\rho(S_1, > 0))^{\mathcal{I}_{\mathcal{C}}} \cap (\rho(S_2, > 0))^{\mathcal{I}_{\mathcal{C}}} = \emptyset \equiv \mathcal{I}_{\mathcal{C}} \models (dis(\rho(S_1, > 0), \rho(S_2, > 0))) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(dis(S_1, S_2))$.

(7) $\tau$ is a reflexive role axiom. Assume that $\mathcal{I} \models ref(R)$. Then, $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, x) = 1$. By induction hypothesis, $\forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, (x, x) \in (\rho(R, \geq 1))^{\mathcal{I}_{\mathcal{C}}} \equiv \forall x \in \Delta^{\mathcal{I}_{\mathcal{C}}}, \mathcal{I}_{\mathcal{C}} \models (x, x) : \rho(R, \geq 1) \equiv \mathcal{I}_{\mathcal{C}} \models \kappa(ref(R))$.

(8) $\tau$ is an asymmetry role axiom. Assume that $\mathcal{I} \models asy(S)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x, y) > 0$ then $S^{\mathcal{I}}(y, x) = 0$. By induction hypothesis, $\forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, if $(x, y) \in (\rho(S, > 0))^{\mathcal{I}_{\mathcal{C}}}$ then $(y, x) \in (\rho(S, \leq 0))^{\mathcal{I}_{\mathcal{C}}} \equiv \forall x, y \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, if $(x, y) \in (\rho(S, > 0))^{\mathcal{I}_{\mathcal{C}}}$ then $(y, x) \notin (\rho(S, > 0))^{\mathcal{I}_{\mathcal{C}}}$. Consequently, $\mathcal{I}_{\mathcal{C}} \models \kappa(asy(\rho(S, > 0)))$.

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If $\mathcal{K}(fK)$ is satisfiable, it is not possible (due to the axioms in $T(N^{fK})$) to have an individual $a$ such that $a^{\mathcal{I}_{\mathcal{C}}} \in (A_{\rhd\gamma_1})^{\mathcal{I}_{\mathcal{C}}}$ and $a^{\mathcal{I}_{\mathcal{C}}} \notin (A_{\rhd\gamma_2})^{\mathcal{I}_{\mathcal{C}}}$ with $\gamma_2 < \gamma_1$, so for every individual $a$ we can compute the maximum value $\alpha$ such that $a : A_{\geq\alpha}$ holds, or the maximum value $\beta$ such that $a : A_{>\beta}$ holds, and use these values in the construction of the fuzzy interpretation. The case for roles in $R_a(N^{fK})$ and $R_c(N^{fK})$ is similar.