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Fuzzy description logics under Gödel semantics

Fernando Bobillo [a,*], Miguel Delgado [b], Juan Gómez-Romero [b], Umberto Straccia [c]

[a] Department of Computer Science and Systems Engineering, University of Zaragoza, Granada, Spain
[b] Department of Computer Science and Artificial Intelligence, University of Granada, Spain
[c] Istituto di Scienza e Tecnologie dell'Informazione (ISTI), Consiglio Nazionale delle Ricerche (CNR), Pisa, Italy

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ABSTRACT

Classical ontologies are not suitable to represent vague pieces of information, which has lead to the birth of Fuzzy Description Logics as an appropriate formalism to represent this type of knowledge. Different families of fuzzy operators lead to Fuzzy Description Logics with different properties. This paper studies Fuzzy Description Logics under a semantics given by the Gödel family of fuzzy operators. We investigate some logical properties and show the decidability of a fuzzy extension of the logic $\mathcal{SROIQ}$, theoretical basis of the language OWL 1.1, by providing a reasoning preserving procedure to obtain a crisp representation for it. Additionally, we show how to represent some types of concept and role modifiers.

1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning [1]. An ontology is defined as an explicit and formal specification of a shared conceptualization [2], which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies allow to add semantics to data, making knowledge maintenance, information integration as well as the reuse of components easier.

The current standard language for ontology creation is the Web Ontology Language (OWL [3]), which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full. OWL Full is the most expressive level but reasoning within it becomes undecidable, OWL Lite has the lowest complexity and OWL DL is a balanced tradeoff between expressiveness and reasoning complexity. However, since its first development, several limitations on expressiveness of OWL have been identified [4], and consequently several extensions to the language have been proposed [5]. Among them, the most significant is OWL 1.1 [4,6] which is its most likely immediate successor.

Description Logics (DLs for short) [7] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to be very useful as ontology languages [8]. For instance, OWL Lite, OWL DL and OWL 1.1 are close equivalents to $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SROIQ}(\mathbf{D})$ respectively [9].

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real world domains [10]. Since fuzzy set theory and fuzzy logic are suitable

* Corresponding author.
E-mail addresses: fbobillo@unizar.es (F. Bobillo), mdelgado@ugr.es (M. Delgado), jgomez@decsai.ugr.es (J. Gómez-Romero), straccia@isti.cnr.it (U. Straccia).

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formalisms to handle these types of knowledge, several fuzzy extensions of DLs can be found in the literature [11], yielding fuzzy ontologies. Fuzzy ontologies have proved to be useful in several applications, such as Chinese news summarization [12], semantic help-desk Support [13], ontology-based query enrichment [14], information retrieval [15] or image interpretation, which has for instance been applied to recognition of brain structures in 3D magnetic resonance images [16]. There are also a lot of applications in the Semantic Web field (see for example [17,18]) and, more generally, in the Internet [10].

It is well known that different families of fuzzy operators lead to fuzzy DLs with different properties [19]. In fuzzy logic, there are three main families of fuzzy operators: Łukasiewicz, Gödel and Product. Nevertheless, most of the previous works rely on the semantics of fuzzy set operators proposed by Zadeh: Gödel conjunction and disjunction, Łukasiewicz negation and Kleene-Dienes implication (see Section 2.2 for the definition of these fuzzy operators and families). Some few works consider Łukasiewicz family, but Gödel family has not received such attention (see Section 6 for a longer discussion).

In our opinion, the logical properties of Gödel family make interesting its study. For example, as well as Zadeh family, Gödel family includes an idempotent conjunction (minimum) so the conjunction is independent of the granularity of the fuzzy ontology (for example, $\min\{0.5, 0.5, 0.5, 0.5\} = \min\{0.5, 0.5\}$), which is interesting in some applications. This is not the case in Łukasiewicz or Product families. But an important difference with respect to Zadeh family is that Gödel family has an R-implication with better logical properties than Kleene-Dienes implication. As it has been pointed out recently [20], Kleene-Dienes implication has some counter-intuitive effects. For example, concepts and roles do not fully subsume themselves.

In this paper, we define a fuzzy extension of the DL $\mathcal{SROIQ}$ under Gödel semantics and show the decidability by providing a reasoning algorithm based on a reduction to a crisp DL. We focus on the very expressive logic $\mathcal{SROIQ}$ because it is the theoretical counterpart of OWL 1.1, a serious candidate to become the next standard language for ontology representation. Additionally, we show how to represent some constructors which are independent from the family of fuzzy operators considered: modified concept and roles.

The remainder of this paper is organized as follows. The following section reviews some background on DLs and fuzzy logic. Then, Section 3 describes a fuzzy extension of $\mathcal{SROIQ}$ under Gödel semantics and discusses some logical properties. Section 4 depicts a reduction of fuzzy $\mathcal{SROIQ}$ into crisp $\mathcal{SROIQ}$, leaving the reduction of modifiers to Section 5. Section 6 reviews some related work and, finally, in Section 7 we set out some conclusions and ideas for future research.

2. Preliminaries

This section provides some background. Section 2.1 describes $\mathcal{SROIQ}$ [21], the DL which will be mainly treated throughout this paper. Section 2.2 refreshes some basic ideas in fuzzy set theory and fuzzy logic [22,23].

2.1. The Description Logic $\mathcal{SROIQ}$

$\mathcal{SROIQ}$ extends $\mathcal{ALC}$ standard DL [24] with transitive roles ($\mathcal{ALC}$ plus transitive roles is called $\mathcal{S}$), complex role axioms ($\mathcal{R}$), nominals ($\mathcal{O}$), inverse roles ($\mathcal{I}$) and qualified number restrictions ($\mathcal{Q}$).

Syntax. $\mathcal{SROIQ}$ assumes three alphabets of symbols, for concepts, roles and individuals. In DLs, complex concepts and roles can be built using different concept and role constructors. In $\mathcal{SROIQ}$, the concepts (denoted C or D) and roles (R) can be built inductively from atomic concepts (A), atomic roles (R_A), top concept $\top$, bottom concept $\bot$, named individuals (o_i), simple roles (S, which will be defined below) and universal role U, as shown in Table 1, where n, m are natural numbers ($n \geqslant 0, m > 0$), $x, y \in \Delta^{\mathcal{I}}$ are abstract individuals and $\sharp X$ denotes the cardinality of the set X.

Table 1
Syntax and semantics of the Description Logic $\mathcal{SROIQ}$.

Constructor	Syntax	Semantics
(Atomic concept)	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
(Top concept)	$\top$	$\Delta^{\mathcal{I}}$
(Bottom concept)	$\bot$	$\emptyset$
(Concept conjunction)	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
(Concept disjunction)	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
(Concept negation)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
(Universal quantification)	$\forall R.C$	$\{x \mid \forall y, (x,y) \notin R^{\mathcal{I}} \text{or} y \in C^{\mathcal{I}}\}$
(Existential quantification)	$\exists R.C$	$\{x \mid \exists y, (x,y) \in R^{\mathcal{I}} \text{and} y \in C^{\mathcal{I}}\}$
(Nominals)	$\{o_1, \ldots, o_m\}$	$\{o_1^{\mathcal{I}}, \ldots, o_m^{\mathcal{I}}\}$
(At least number restriction)	$\geqslant n\ S.C$	$\{x \mid \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{and} y \in C^{\mathcal{I}}\} \geqslant n\}$
(At-most number restriction)	$\leqslant n\ S.C$	$\{x \mid \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{and} y \in C^{\mathcal{I}}\} \leqslant n\}$
(Local reflexivity)	$\exists S.\mathtt{Self}$	$\{x \mid (x,x) \in S^{\mathcal{I}}\}$
(Atomic role)	R_A	$R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
(Inverse role)	R^-	$\{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\}$
(Universal role)	U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

Example 2.1. $\texttt{Man}$ and $\texttt{Woman}$ are atomic concepts. $\texttt{hasChild}$ and $\texttt{likes}$ are atomic roles. $\texttt{Man} \sqcap \geqslant 2 \texttt{hasChild}.\texttt{Woman}$ is a complex concept representing a father with at least two daughters. $\exists \texttt{likes}.\texttt{Self}$ represents a narcisist.

A Knowledge Base (KB) comprises the intensional knowledge, i.e. general knowledge about the application domain (a Terminological Box or *TBox* $\mathcal{T}$ and a Role Box or *RBox* $\mathcal{R}$), and the extensional knowledge, i.e. particular knowledge about some specific situation (an Assertional Box or *ABox* $\mathcal{A}$ with statements about individuals).

An ABox consists of a finite set of *assertions* about individuals:

- *concept assertions* $a : C$, meaning that individual a is an instance of C,
- *role assertions* $(a, b) : R$, meaning that (a, b) is an instance of R,
- *negated role assertions* $(a, b) : \neg R$,
- *inequality assertions* $a \neq b$,
- *equality assertions* $a = b$.

A TBox consists of a finite set of *general concept inclusion* (*GCI*) axioms $C \sqsubseteq D$ (C is more specific than D).

Let w be a role chain (a finite string of roles not including the universal role U). An RBox consists of a finite set of role axioms:

- *role inclusion axioms (RIAs)* $w \sqsubseteq R$ (role chain w is more specific than R),
- *transitive role axioms* $\texttt{trans}(R)$,
- *disjoint role axioms* $\texttt{dis}(S_1, S_2)$,
- *reflexive role axioms* $\texttt{ref}(R)$,
- *irreflexive role axioms* $\texttt{irr}(S)$,
- *symmetric role axioms* $\texttt{sym}(R)$,
- *asymmetric role axioms* $\texttt{asy}(S)$.

Example 2.2. The concept assertion $\texttt{paul} : \texttt{Man}$ states that the individual Paul belongs to the class of men. The role assertion $(\texttt{paul}, \texttt{john}) : \neg \texttt{hasChild}$ states that John is not the child of Paul. The GCI $\texttt{Man} \sqsubseteq \texttt{Human}$ states that all men are human. The RIA $\texttt{owns}\ \texttt{hasPart} \sqsubseteq \texttt{owns}$ states the fact if somebody owns something, he also owns its components.

Now we will introduce some definitions which will be useful to impose some limitations in the language. A *strict partial order* $\prec$ on a set A is an irreflexive and transitive relation on A. A strict partial order $\prec$ on the set of roles is called a *regular order* if it also satisfies $R_1 \prec R_2 \Longleftrightarrow R_2^- \prec R_1$, for all roles R_1 and R_2.

In order to guarantee the decidability of the logic, there are some restrictions in the use of roles. Given a regular order $\prec$, every role axiom cannot contain U and every RIA should be $\prec$-regular. A RIA $w \sqsubseteq R$ is $\prec$*-regular* if R is atomic and:

1. $w = RR$, or
2. $w = R^-$, or
3. $w = S_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
4. $w = RS_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
5. $w = S_1 \ldots S_n R$ and $S_i \prec R$ for all $i = 1, \ldots, n$.

Note that, in order to prove decidability of the reasoning, roles are assumed to be simple in some concept constructors (local reflexivity, at least and at-most number restrictions) and role axioms (disjoint, irreflexive and asymmetric role axioms) [21]. *Simple* roles are defined as follows:

1. R_A is simple if it does not occur on the right side of a RIA,
2. R^- is simple if R is,
3. if R occurs on the right side of a RIA, R is simple if, for each $w \sqsubseteq R$, $w = S$ for a simple role S.

Semantics. An interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and an interpretation function $\cdot^{\mathcal{I}}$ mapping:

- every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$,
- every atomic concept A onto a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$,
- every atomic role R_A onto a relation $R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$,

The interpretation is extended to complex concepts and roles by the inductive definitions in Table 1. Unique name assumption is not imposed, i.e. two nominals might refer to the same individual.

Let $\circ$ be the standard composition of relations. An interpretation $\mathcal{I}$ satisfies (is a model of):

- $a : C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $(a, b) : R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$,
- $(a, b) : \neg R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$,
- $a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $R_1 \ldots R_n \sqsubseteq R$ iff $R_1^{\mathcal{I}} \circ \ldots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$,
- $\mathtt{trans}(R)$ iff $(x, y) \in R^{\mathcal{I}}$ and $(y, z) \in R^{\mathcal{I}}$ imply $(x, z) \in R^{\mathcal{I}}$, $\forall x, y, z \in \Delta^{\mathcal{I}}$,
- $\mathtt{dis}(S_1, S_2)$ iff $S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$,
- $\mathtt{ref}(R)$ iff $(x, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $\mathtt{irr}(S)$ iff $(x, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $\mathtt{sym}(R)$ iff $(x, y) \in R^{\mathcal{I}}$ implies $(y, x) \in R^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- $\mathtt{asy}(S)$ iff $(x, y) \in S^{\mathcal{I}}$ implies $(y, x) \notin S^{\mathcal{I}}, \forall x \in \Delta^{\mathcal{I}}$,
- a KB $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

A DL not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability, concept satisfiability or subsumption. However, if a DL is closed under negation, most of the basic reasoning tasks are reducible to KB satisfiability [25], so it is usually the only task considered.

2.2. Fuzzy Set Theory

Fuzzy set theory and fuzzy logic were proposed by Zadeh [26] to manage imprecise and vague knowledge. While in classical set theory elements either belong to a set or not, in fuzzy set theory elements can belong to a set to some degree. More formally, let X be a set of elements called the reference set. A fuzzy subset A of X is defined by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns any $x \in X$ to a value in the interval of real numbers between 0 and 1. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which x can be considered as an element of X. If the reference set is finite ($X = \{x_1, \ldots, x_n\}$), the membership function can be expressed using the notation $A = \{\mu_A(x_1)/x_1, \ldots \mu_A(x_n)/x_n\}$.

For every $\alpha \in [0, 1]$, the *α-cut* of a fuzzy set A is defined as the (crisp) set such that its elements belong to A with degree at least α, i.e. $\{x | \mu_A(x) \geqslant \alpha\}$. Similarly, the *strict α-cut* is defined as $\{x | \mu_A(x) > \alpha\}$.

A fuzzy *modifier* is a function $f : [0, 1] \to [0, 1]$ which is applied to a fuzzy set in order to change its membership function. For example, the modifier "very" is sometimes defined as $f_{very}(x) = x^2$.

All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication set operations are performed by a t-norm function, a t-conorm function, a negation function and an implication function, respectively. Table 2 shows the most important families of fuzzy operators: Zadeh, Łukasiewicz, Gödel and Product.

The operation of fuzzy intersection is performed by a *t-norm* function $\otimes : [0, 1] \times [0, 1] \to [0, 1]$, i.e., a function satisfying the following properties: (i) *boundary condition* i.e., $\alpha \otimes 1 = \alpha$; ($ii$) *increasing monotonicity* i.e., for each $\beta \leqslant \gamma$ then $\alpha \otimes \beta \leqslant \alpha \otimes \gamma$; ($iii$) *commutativity* i.e., $\alpha \otimes \beta = \beta \otimes \alpha$; ($iv$) *associativity* i.e., $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$. Every t-norm satisfies $\alpha, \beta \geqslant \alpha \otimes \beta$ and $\alpha \otimes 0 = 0$.

Fuzzy union is performed by a *t-conorm* (or s-norm) function $\oplus : [0, 1] \times [0, 1] \to [0, 1]$, i.e., a function satisfying: (i) *boundary condition* i.e., $\alpha \oplus 0 = \alpha$; (ii) *increasing monotonicity* i.e., for each $\beta \leqslant \gamma$ then $\alpha \oplus \beta \leqslant \alpha \oplus \gamma$; ($iii$) *commutativity* i.e., $\alpha \oplus \beta = \beta \oplus \alpha$; ($iv$) *associativity* i.e., $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$. Every t-conorm satisfies $\alpha, \beta \leqslant \alpha \oplus \beta$, and $\alpha \oplus 1 = 1$.

Fuzzy complement is performed by a *negation* function $\ominus : [0, 1] \to [0, 1]$ satisfying: (i) *boundary conditions* i.e., $\ominus 0 = 1$ and $\ominus 1 = 0$; (ii) *decreasing monotonicity* i.e., for each $\alpha \leqslant \beta, \ominus\alpha \geqslant \ominus\beta$. Gödel negation is *discontinuous* and *non-involutive* i.e., in general, $\ominus(\ominus\alpha) \neq \alpha$.

Fuzzy implication is performed by an *implication* function $\Rightarrow: [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following properties: (i) *monotonicity* i.e., $\alpha \leqslant \beta$ implies $(\alpha \Rightarrow \gamma) \geqslant (\beta \Rightarrow \gamma)$ and $\beta \leqslant \gamma$ implies $\alpha \Rightarrow \beta \leqslant \alpha \Rightarrow \gamma$; ($ii$) *boundary conditions* i.e., $0 \Rightarrow \alpha = \alpha \Rightarrow 1 = 1$ and $1 \Rightarrow 0 = 0$.

Table 2
Popular families of fuzzy operators.

Family	t-Norm $\alpha \otimes \beta$	t-Conorm $\alpha \oplus \beta$	Negation $\ominus\alpha$	Implication $\alpha \Rightarrow \beta$
Zadeh	$\min\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$1 - \alpha$	$\max\{1 - \alpha, \beta\}$
Łukasiewicz	$\max\{\alpha + \beta - 1, 0\}$	$\min\{\alpha + \beta, 1\}$	$1 - \alpha$	$\min\{1 - \alpha + \beta, 1\}$
Gödel	$\min\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$	$\begin{cases} 1 & \alpha \leqslant \beta \\ \beta, & \alpha > \beta \end{cases}$
Product	$\alpha \cdot \beta$	$\alpha + \beta - \alpha \cdot \beta$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$	$\begin{cases} 1 & \alpha \leqslant \beta \\ \beta/\alpha, & \alpha > \beta \end{cases}$

There are commonly two types of fuzzy implications used. The first class is *S-implications*, which extend the crisp proposition $\alpha \Rightarrow b = \neg\alpha \vee b$ to the fuzzy case and are defined by the operation $\alpha \Rightarrow \beta = (\ominus\alpha) \oplus \beta$. The second class is *R-implications* (residuum-based implications), which are defined as $\alpha \Rightarrow \beta = \sup\{\gamma \in [0,1] | (\alpha \otimes \gamma) \leqslant \beta\}$ and can be used to define a fuzzy complement as $\ominus a = a \Rightarrow 0$. They always verify that $\alpha \Rightarrow \beta = 1$ iff $\alpha \leqslant \beta$. Furthermore, they allow to apply modus ponens in the following way. If a proposition ψ is true to degree α and $\psi \Rightarrow \phi$ is true to degree β, then ϕ is true to degree $\alpha \otimes \beta$, where $\otimes$ is the t-norm associated to $\Rightarrow$. Product and Gödel implications are R-implications, the implication of the Zadeh family which is called Kleene-Dienes (KD) is an S-implication, and the Łukasiewicz implication belongs to both types.

A fuzzy set C is *included* in another fuzzy set D iff $\forall x \in X, \mu_C(x) \leqslant \mu_D(x)$. According to this definition, which is usually called Zadeh's set inclusion, fuzzy set inclusion is a yes–no question. In order to overcome this, other definitions have been proposed. For example, the degree of inclusion of C in D can be computed using some implication function as $\inf_{x \in X} \mu_C(x) \Rightarrow \mu_D(x)$. In this paper we will follow the latter approach.

3. Fuzzy $\mathcal{SROIQ}$

In this section we define f-$\mathcal{SROIQ}$, which extends $\mathcal{SROIQ}$ to the fuzzy case by letting (*i*) concepts denote fuzzy sets of individuals and (*ii*) roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. The following definition extends [27,28] with fuzzy nominals [20] and cut concepts and roles [29].

3.1. Definition

In the rest of the paper we will assume $\bowtie \in \{\geqslant, <, \leqslant, >\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$ and $\gamma \in [0,1]$. The symmetric $\bowtie^-$ and the negation $\neg \bowtie$ of an operator $\bowtie$ are defined as follows:

$\bowtie$	$\bowtie^-$	$\neg \bowtie$
$\geqslant$	$\leqslant$	$<$
$>$	$<$	$\leqslant$
$\leqslant$	$\geqslant$	$>$
$<$	$>$	$\geqslant$

Syntax. f-$\mathcal{SROIQ}$ assumes three alphabets of symbols, for concepts, roles and individuals. Let U be the universal role, R_A an atomic role and *mod* a fuzzy modifier. The roles of the language are built using the syntax rule[1]:

$$R \rightarrow R_A | U | R^- | mod(R) | [R \geqslant \alpha] \tag{1}$$

The concepts of the language (denoted C or D) can be built inductively from atomic concepts (A), top concept $\top$, bottom concept $\bot$, named individuals (o_i) and roles (R and S, where S is a simple role as defined below) according to the following syntax rule (with n, m being natural numbers, $n \geqslant 0, m > 0$):

$$\begin{aligned} C, D \rightarrow\ & A | \top | \bot | C \sqcap D | C \sqcup D | \neg C | \forall R.C | \exists R.C | \\ & \{\alpha_1/o_1, \ldots \alpha_m/o_m\} | (\geqslant m\ S.C) | (\leqslant n\ S.C) | \\ & \exists S.\mathtt{Self} |\ mod(C) |\ [C \geqslant \alpha] \end{aligned} \tag{2}$$

The only differences with the crisp case are fuzzy nominals $\{\alpha_1/o_1, \ldots \alpha_m/o_m\}$ [20], concept and role modifiers $mod(C), mod(R)$ [30] and cut concepts and roles $[C \geqslant \alpha], [R \geqslant \alpha]$ [29].

Example 3.1. $\{1/\mathtt{germany}, 1/\mathtt{austria}, 0.67/\mathtt{switzerland}\}$ represents the concept of German-speaking country, with Germany and Austria fully belonging to it, but Switzerland belonging only with degree 0.67. $very(\mathtt{Tall})$ represents the fuzzy set of individuals which are very tall. $[\mathtt{isFriendOf} \geqslant 0.8]$ represents the pairs of individuals which are friends at least to degree 0.8.

A fuzzy KB $\mathcal{K}$ comprises a fuzzy ABox $\mathcal{A}$, a fuzzy TBox $\mathcal{T}$ and a fuzzy RBox $\mathcal{R}$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions*. A fuzzy assertion can be an inequality assertion $\langle a \neq b \rangle$, an equality assertion $\langle a = b \rangle$ or a constraint on the truth value of a concept or role assertion, i.e., an expression of the form $\langle \Psi \geqslant \alpha \rangle$, $\langle \Psi > \beta \rangle$, $\langle \Psi \leqslant \beta \rangle$ or $\langle \Psi < \alpha \rangle$, where Ψ is of the form $a : C$, $(a,b) : R$ or $(a,b) : \neg R$.

A *fuzzy TBox* consists of *fuzzy GCIs*, which constrain the truth value of a GCI i.e. they are expressions of the form $\langle C \sqsubseteq D \geqslant \alpha \rangle$ or $\langle C \sqsubseteq D > \beta \rangle$.

A *fuzzy RBox* consists of a finite set of role axioms, which can be *fuzzy RIAs* $\langle w \sqsubseteq R \geqslant \alpha \rangle$ or $\langle w \sqsubseteq R > \beta \rangle$ for a role chain $w = R_1 R_2 \ldots R_n$, or any other of the role axioms from the crisp case: *transitive* $\mathtt{trans}(R)$, *disjoint* $\mathtt{dis}(S_1, S_2)$, *reflexive* $\mathtt{ref}(R)$, *irreflexive* $\mathtt{irr}(S)$, *symmetric* $\mathtt{sym}(R)$ or *asymmetric* $\mathtt{asy}(S)$.

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1 We will also allow role negation in fuzzy assertions of the form $\langle (a,b) : \neg R \bowtie \alpha \rangle$.

Example 3.2. The fuzzy concept assertion $\langle \texttt{paul} : \texttt{Tall} \geqslant 0.5\rangle$ states that Paul is tall with at least degree 0.5. The fuzzy RIA $\langle \texttt{isFriendOf isFriendOf} \sqsubseteq \texttt{isFriendOf} \geqslant 0.75\rangle$ states that the friends of my friends can also my considered as my friends with at least degree 0.75.

A fuzzy axiom is *positive* (denoted $\langle \tau \rhd \alpha\rangle$) if it is of the form $\langle \tau \geqslant \alpha\rangle$ or $\langle \tau > \beta\rangle$, and *negative* (denoted $\langle \tau \lhd \alpha\rangle$) if it is of the form $\langle \tau \leqslant \beta\rangle$ or $\langle \tau < \alpha\rangle$. $\langle \tau = \alpha\rangle$ is equivalent to the pair of axioms $\langle \tau \geqslant \alpha\rangle$ and $\langle \tau \leqslant \alpha\rangle$ [31]. Of course, $\tau \equiv \langle \tau \geqslant 1\rangle$.

Notice that negative fuzzy GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs respectively, which are not part of crisp $\mathcal{SROIQ}$.

As in the crisp case, role axioms cannot contain U and every RIA should be $\prec$-regular for a regular order $\prec$. A RIA $\langle w \sqsubseteq R \rhd \gamma\rangle$ is *$\prec$-regular* if R is atomic and: (*i*) $w = RR$, or (*ii*) $w = R^-$, or (*iii*) $w = S_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1,\ldots,n$, or (*iv*) $w = RS_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1,\ldots,n$, or (*v*) $w = S_1 \ldots S_n R$ and $S_i \prec R$ for all $i = 1,\ldots,n$.

Simple roles are defined as in the crisp case: (*i*) R_A is simple if it does not occur on the right side of a RIA, (*ii*) R^- is simple if R is, (*iii*) if R occurs on the right side of a RIA, R is simple if, for each $\langle w \sqsubseteq R \rhd \gamma\rangle$, $w = S$ for a simple role S.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of a non empty set $\Delta^\mathcal{I}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^\mathcal{I}$ mapping:

- every individual a onto an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$,
- every concept C onto a function $C^\mathcal{I} : \Delta^\mathcal{I} \to [0,1]$,
- every role R onto a function $R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$,
- every modifier *mod* onto a function $f_{mod} : [0,1] \to [0,1]$,

$C^\mathcal{I}$ (resp. $R^\mathcal{I}$) denotes the membership function of the fuzzy concept C (resp. fuzzy role R) w.r.t. $\mathcal{I}$. $C^\mathcal{I}(a)$ (resp. $R^\mathcal{I}(a,b)$) gives us to what extent the individual a can be considered as an element of the fuzzy concept C (resp. to what extent (a,b) can be considered as an element of the fuzzy role R) under the fuzzy interpretation $\mathcal{I}$.

Given a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

$$
\begin{aligned}
&\top^\mathcal{I}(x) = 1\\
&\bot^\mathcal{I}(x) = 0\\
&(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)\\
&(C \sqcup D)^\mathcal{I}(x) = C^\mathcal{I}(x) \oplus D^\mathcal{I}(x)\\
&(\neg C)^\mathcal{I}(x) = \ominus C^\mathcal{I}(x)\\
&(\forall R.C)^\mathcal{I}(x) = \inf_{y \in \Delta^\mathcal{I}} \{R^\mathcal{I}(x,y) \Rightarrow C^\mathcal{I}(y)\}\\
&(\exists R.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} \{R^\mathcal{I}(x,y) \otimes C^\mathcal{I}(y)\}\\
&\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}^\mathcal{I}(x) = \sup_{i \mid x = o_i^\mathcal{I}} \alpha_i\\
&(\geqslant m\ S.C)^\mathcal{I}(x) = \sup_{y_1,\ldots,y_m \in \Delta^\mathcal{I}} [(\otimes_{i=1}^m \{S^\mathcal{I}(x,y_i) \otimes C^\mathcal{I}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\})]\\
&(\leqslant n\ S.C)^\mathcal{I}(x) = \inf_{y_1,\ldots,y_{n+1} \in \Delta^\mathcal{I}} [(\otimes_{i=1}^{n+1} \{S^\mathcal{I}(x,y_i) \otimes C^\mathcal{I}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})]\\
&(\exists S.\texttt{Self})^\mathcal{I}(x) = S^\mathcal{I}(x,x)\\
&(mod(C))^\mathcal{I}(x) = f_{mod}(C^\mathcal{I}(x))\\
&([C \geqslant \alpha])^\mathcal{I}(x) = 1 \text{ if } C^\mathcal{I}(x) \geqslant \alpha, 0 \quad \text{otherwise}\\
&(R^-)^\mathcal{I}(x,y) = R^\mathcal{I}(y,x)\\
&U^\mathcal{I}(x,y) = 1\\
&(mod(R))^\mathcal{I}(x,y) = f_{mod}(R^\mathcal{I}(x,y))\\
&([R \geqslant \alpha])^\mathcal{I}(x,y) = 1 \text{if} R^\mathcal{I}(x,y) \geqslant \alpha, 0 \quad \text{otherwise}
\end{aligned}
$$

We do not impose unique name assumption, i.e. two nominals might refer to the same individual. Note that cut concepts are crisp, as opposed to [32,33].

The fuzzy interpretation function is extended to fuzzy axioms as follows:

$$
\begin{aligned}
&(a : C)^\mathcal{I} = C^\mathcal{I}(a^\mathcal{I})\\
&((a,b) : R)^\mathcal{I} = R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})\\
&((a,b) : \neg R)^\mathcal{I} = \ominus R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})\\
&(C \sqsubseteq D)^\mathcal{I} = \inf_{x \in \Delta^\mathcal{I}} C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)\\
&(R_1 \ldots R_n \sqsubseteq R)^\mathcal{I} = \sup_{x_1 \ldots x_{n+1} \in \Delta^\mathcal{I}} \otimes[R_1^\mathcal{I}(x_1,x_2), \ldots, R_n^\mathcal{I}(x_n,x_{n+1})] \Rightarrow R^\mathcal{I}(x_1,x_{n+1})
\end{aligned}
$$

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- $\langle a : C \bowtie \gamma \rangle$ iff $(a : C)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a,b) : R \bowtie \gamma \rangle$ iff $((a,b) : R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a,b) : \neg R \bowtie \gamma \rangle$ iff $((a,b) : \neg R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle a \neq b \rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $\langle a = b \rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $\langle C \sqsubseteq D \rhd \gamma \rangle$ iff $(C \sqsubseteq D)^{\mathcal{I}} \rhd \gamma$,
- $\langle R_1 \ldots R_n \sqsubseteq R \rhd \gamma \rangle$ iff $(R_1 \ldots R_n \sqsubseteq R)^{\mathcal{I}} \rhd \gamma$,
- $\mathtt{trans}(R)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geqslant \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$,
- $\mathtt{dis}(S_1, S_2)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = 0$ or $S_2^{\mathcal{I}}(x,y) = 0$,
- $\mathtt{ref}(R)$ iff $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$,
- $\mathtt{irr}(S)$ iff $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$,
- $\mathtt{sym}(R)$ iff $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$,
- $\mathtt{asy}(S)$ iff $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$,
- a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

Notice that individual assertions are considered to be crisp, since the equality and inequality of individuals has always been considered crisp in the fuzzy DL literature [27,34].

In the rest of the paper we will only consider fuzzy KB satisfiability, since (as in the crisp case) most inference problems can be reduced to it [35].

Example 3.3. The following tasks can be reduced to fuzzy KB satisfiability:

- *Concept satisfiability*. C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle x : C \geqslant \alpha \rangle\}$ is satisfiable, where x is a new individual, which does not appear in $\mathcal{K}$.
- *Entailment*: A fuzzy concept assertion $a : C \bowtie \alpha$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \bowtie \alpha \rangle$) iff $\mathcal{K} \cup \{\langle a : C \neg \bowtie \alpha \rangle\}$ is unsatisfiable. The case for fuzzy role assertions is similar.
- *Greatest lower bound*. The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle \tau \geqslant \alpha \rangle\}$. In Łukasiewicz, Zadeh and Gödel families, it can be computed performing several entailment tests.[2]

Finally, in order to manage correctly infima and suprema in the reasoning, we need to define the notion of *witnessed* interpretations. A fuzzy interpretation $\mathcal{I}$ is *witnessed* [36] iff it verifies:

- for all $x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\exists R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$, and
- for all $x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\forall R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$, and
- there is $x \in \Delta^{\mathcal{I}}$ such that $(C \sqsubseteq D)^{\mathcal{I}} = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$, and
- there are $x_1, \ldots, x_{n+1} \in \Delta^{\mathcal{I}}$ such that $(R_1 \ldots R_n \sqsubseteq R)^{\mathcal{I}} = (R_1^{\mathcal{I}}(x_1,x_2) \otimes \ldots \otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})$, and
- if $\mathcal{I} \models \mathtt{trans}(R)$, for all $x, y \in \Delta^{\mathcal{I}}$, there is $z \in \Delta^{\mathcal{I}}$ such that $\sup_{z' \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z') \otimes R^{\mathcal{I}}(z',y) = R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$.

3.2. Logical properties

It can be easily shown that f-$\mathcal{SROIQ}$ is a sound extension of crisp $\mathcal{SROIQ}$, in the sense that fuzzy interpretations coincide with crisp interpretations if we restrict the degrees of truth to $\{0,1\}$.

In the rest of the paper we will concentrate on $G\mathcal{SROIQ}$, restricting ourselves to the Gödel family.

In general, Gödel logic does not have the witnessed model property, i.e. there can exist fuzzy KBs which have an infinite model, but they do not have a witnessed model (see [36] for an example). However, due to the limited precision of computers, we will restrict to a finite set $\mathcal{TV}$. Given a fuzzy KB $\mathcal{K}$, we will also assume that $\mathcal{TV}$ includes at least every degree in $\mathcal{K}$ plus 0 and 1. It can be shown that in Gödel logic over a fixed finite set of degrees of truth including 0 and 1, all models (finite or infinite) are witnessed [36]. Hence, in our logic every interpretation $\mathcal{I}$ is witnessed.

Due to the standard properties of the fuzzy operators, the following concept equivalences hold [35]: $\neg\top \equiv \bot$, $\neg\bot \equiv \top$, $C \sqcap \top \equiv C$, $C \sqcup \bot \equiv C$, $C \sqcap \bot \equiv \bot$, $C \sqcup \top \equiv \top$, $\exists R.\bot = \bot$, $\forall R.\top = \top$.

Moreover, the choice of the fuzzy operators implies the following properties:

1. Negation is not *involutive*: $\neg\neg C \not\equiv C$.
2. Law of *excluded middle* does not hold: $C \sqcup \neg C \not\equiv \top$.
3. Law of *contradiction* holds: $C \sqcap \neg C \equiv \bot$.

[2] More precisely, in Gödel logic we need to assume a finite set of degrees of truth $\mathcal{TV}$ including 0 and 1 [36], and can be computed performing at-most $log|\mathcal{TV}|$ tests [35].

4. *Idempotent* conjunction and disjunction: $C \sqcap C \equiv C$ and $C \sqcup C \equiv C$.
5. *De Morgan* laws: $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$ and $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$. However, $C \sqcup D \not\equiv \neg(\neg C \sqcap \neg D)$ and $C \sqcap D \not\equiv \neg(\neg C \sqcup \neg D)$.
6. *Non inter-definability of quantifiers*: $\forall R.C \not\equiv \neg\exists R.(\neg C)$ and $\exists R.C \not\equiv \neg\forall R.(\neg C)$. Moreover, $\neg\forall R.C \not\equiv \exists R.(\neg C)$ but $\neg\exists R.C \equiv \forall R.(\neg C)$.
7. *Non inter-definability of qualified cardinality restrictions*: $(\geqslant m\ S.C) \not\equiv \neg(\leqslant m-1\ S.C)$, but $(\leqslant n\ S.C) \equiv \neg(\geqslant n+1\ S.C)$.

Properties 1–5 follow immediately from the semantics of the fuzzy operators. Although in general quantifiers and qualified cardinality restrictions are not inter-definable, the following proposition shows that two interesting equivalences hold.

Proposition 3.1. *Under $\mathcal{GSROIQ}$ the following properties hold:*

1. $\neg\exists R.C \equiv \forall R.(\neg C)$
2. $(\leqslant n\ S.C) \equiv \neg(\geqslant n+1\ S.C)$

In crisp DLs, the assertion $a : C$ is equivalent to the GCI $\{a\} \sqsubseteq C$. This can be extended to the fuzzy case, as the following proposition shows:

Proposition 3.2. *In fuzzy $\mathcal{SROIQ}$ under an R-implication, the following equivalence holds:*

$$\langle a : C \geqslant \alpha\rangle \equiv \langle\{\alpha/a\} \sqsubseteq C \geqslant 1\rangle$$

Similarly as in Zadeh logic [37], $\mathcal{GSROIQ}$ allows some sort of *modus ponens* and *chaining* of GCIs and RIAs:

Proposition 3.3. *For $\alpha, \beta \in [0,1]$ and $\triangleright \in \{\geqslant, >\}$, the following properties are verified:*

(*i*) $\langle a : C \triangleright \alpha\rangle$ *and* $\langle C \sqsubseteq D \triangleright \beta\rangle$ *imply* $\langle a : D \triangleright \min\{\alpha, \beta\}\rangle$.
(*ii*) $\langle (a,b) : R \triangleright \alpha\rangle$ *and* $\langle R \sqsubseteq R' \triangleright \beta\rangle$ *imply* $\langle (a,b) : R' \triangleright \min\{\alpha, \beta\}\rangle$.
(*iii*) $\langle C \sqsubseteq D \triangleright \alpha\rangle$ *and* $\langle D \sqsubseteq E \triangleright \beta\rangle$ *imply* $\langle C \sqsubseteq E \triangleright \min\{\alpha, \beta\}\rangle$.
(*iv*) $\langle R \sqsubseteq R' \triangleright \alpha\rangle$ *and* $\langle R' \sqsubseteq R'' \triangleright \beta\rangle$ *imply* $\langle R \sqsubseteq R'' \triangleright \min\{\alpha, \beta\}\rangle$.

Irreflexive, transitive and symmetric role axioms are syntactic sugar for any R-implication (and consequently it can be assumed that they do not appear in fuzzy KBs) due to some equivalences with fuzzy GCIs and RIAs.

Proposition 3.4. *In fuzzy $\mathcal{SROIQ}$ under an R-implication, the following equivalences hold:*

- $irr(S) \equiv \langle \top \sqsubseteq \neg\exists S.\mathtt{Self} \geqslant 1\rangle$,
- $trans(R) \equiv \langle RR \sqsubseteq R \geqslant 1\rangle$,
- $sym(R) \equiv \langle R \sqsubseteq R^- \geqslant 1\rangle$.

4. An optimized crisp representation for fuzzy $\mathcal{SROIQ}$

In this section we show how to reduce a $\mathcal{GSROIQ}$ fuzzy KB into a crisp KB. The procedure preserves reasoning, so existing $\mathcal{SROIQ}$ reasoners could be applied to the resulting KB. First we will describe the reduction and then we will provide an illustrating example.

The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

4.1. Adding new elements

Let A be the set of atomic concepts and R the set of atomic roles in a fuzzy KB $\mathcal{K} = \langle\mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$. Straccia showed that under Zadeh semantics the set of the degrees which must be considered for any reasoning task are those degrees appearing in the fuzzy KB together with their complementaries. Formally, the set of degrees is defined as $\mathcal{N}^{\mathcal{K}} = \mathcal{X}^{\mathcal{K}} \cup \{1-\gamma | \gamma \in \mathcal{X}^{\mathcal{K}}\}$, where $\mathcal{X}^{\mathcal{K}} = \{0, 0.5, 1\} \cup \{\gamma | \langle\tau \bowtie \gamma\rangle \in \mathcal{K}\}$ [38], where τ is a concept or role assertion, a GCI or a RIA.

The previous property holds for fuzzy DLs under Zadeh semantics, but it is not true in general when other fuzzy operators are considered. Interestingly, in Gödel logic it is enough to consider a fixed set of degrees of truth including 0 and 1 since the fuzzy operators do not introduce new degrees of truth. We define $\mathcal{TV} = \{0,1\} \cup \{\gamma | \langle\tau \bowtie \gamma\rangle \in \mathcal{K}\}$. For every $\gamma_1, \gamma_2 \in \mathcal{TV}$:

- The value of $\ominus\gamma_1$ is either 0 or 1.
- The value of $\gamma_1 \otimes \gamma_2$ and $\gamma_1 \oplus \gamma_2$ is either γ_1 or γ_2.
- The value of $\gamma_1 \Rightarrow \gamma_2$ is either 1 or γ_2.

And, by definition, 0, 1, γ_1 and γ_2 belong to $\mathcal{TV}$. We will also define $\mathcal{TV}^+ = \mathcal{TV} \setminus \{0\}$.

Without loss of generality, it can be assumed that $\mathcal{TV} = \{\gamma_1, \ldots, \gamma_{|\mathcal{TV}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leqslant i \leqslant |\mathcal{TV}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|\mathcal{TV}|} = 1$.

Now, for each $\alpha, \beta \in \mathcal{TV}$ with $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, for each $A \in \mathbf{A}$, two new atomic concepts $A_{\geqslant\alpha}, A_{>\beta}$. are introduced. $A_{\geqslant\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. $A_{>\beta}$ is defined in a similar way. Similarly, for each $R_A \in \mathbf{R}$ two new atomic roles $R_{A\geqslant\alpha}, R_{A>\beta}$. The atomic elements $A_{>1}, R_{A>1}, A_{\geqslant 0}$ and $R_{A\geqslant 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB (e.g. we do not allow GCIs of the form $C \sqsubseteq D \geqslant 0$).

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leqslant i \leqslant |\mathcal{TV}| - 1, 2 \leqslant j \leqslant |\mathcal{TV}| - 1$ and for each $A \in \mathbf{A}$, $T(\mathcal{TV})$ is the smallest terminology containing these two axioms:

$$\begin{aligned} &A_{\geqslant\gamma_{i+1}} \sqsubseteq A_{>\gamma_i} \\ &A_{>\gamma_j} \sqsubseteq A_{\geqslant\gamma_j} \end{aligned} \tag{3}$$

Similarly, for each $R_A \in \mathbf{R}$, $R(\mathcal{TV})$ is the smallest terminology containing:

$$\begin{aligned} &R_{A\geqslant\gamma_{i+1}} \sqsubseteq R_{A>\gamma_i} \\ &R_{A>\gamma_i} \sqsubseteq R_{A\geqslant\gamma_i} \end{aligned} \tag{4}$$

Note that some previous works introduce two more atomic concepts $A_{\leqslant\beta}, A_{<\alpha}$ and several additional axioms [38,20].

$$\begin{aligned} &A_{<\gamma_k} \sqsubseteq A_{\leqslant\gamma_k} \quad A_{\leqslant\gamma_i} \sqsubseteq A_{<\gamma_{i+1}} \\ &A_{\geqslant\gamma_k} \sqcap A_{<\gamma_k} \sqsubseteq \bot \quad A_{>\gamma_i} \sqcap A_{\leqslant\gamma_i} \sqsubseteq \bot \\ &\top \sqsubseteq A_{\geqslant\gamma_k} \sqcup A_{<\gamma_k} \quad \top \sqsubseteq A_{>\gamma_i} \sqcup A_{\leqslant\gamma_i} \end{aligned} \tag{5}$$

In contrast to this, we use $\neg A_{>\gamma_k}$ rather than $A_{\leqslant\gamma_k}$, and $\neg A_{\geqslant\gamma_k}$ instead of $A_{<\gamma_k}$ as proposed in [39]. This way, these six axioms are not necessary since they follow from the semantics of the crisp DL. In the case of roles, we use $\neg R_{A\neg\triangleleft\gamma}$ instead of $R_{A_{\triangleleft\gamma}}$, as we will see in the next subsection. This idea is essential in order to represent some role constructors of $\mathcal{GSROIQ}$ (negated role assertions and self reflexivity concepts). Actually, it is not possible to use a role of the form $R_{A\leqslant\gamma_k}$ rather than $\neg R_{A>\gamma_k}$ and $R_{A<\gamma_k}$ instead of $\neg R_{A\geqslant\gamma_k}$ because the logic does not allow to express the corresponding versions of the four latter axioms in Eq. 5. Having these axioms would be necessary to guarantee the correctness of the reduction, because the role conjunction and the bottom role are not allowed, and the universal role cannot appear in RIAs.

Table 3
Mapping of concept expressions.

x	y	$\rho(x, y)$
$\top$	$\triangleright\gamma$	$\top$
$\top$	$\triangleleft\gamma$	$\bot$
$\bot$	$\triangleright\gamma$	$\bot$
$\bot$	$\triangleleft\gamma$	$\top$
A	$\triangleright\gamma$	$A_{\triangleright\gamma}$
A	$\triangleleft\gamma$	$\neg A_{\neg\triangleleft\gamma}$
$\neg C$	$\triangleright\gamma$	$\rho(C, \leqslant 0)$
$\neg C$	$\triangleleft\gamma$	$\rho(C, > 0)$
$C \sqcap D$	$\triangleright\gamma$	$\rho(C, \triangleright\gamma) \sqcap \rho(D, \triangleright\gamma)$
$C \sqcap D$	$\triangleleft\gamma$	$\rho(C, \triangleleft\gamma) \sqcup \rho(D, \triangleleft\gamma)$
$C \sqcup D$	$\triangleright\gamma$	$\rho(C, \triangleright\gamma) \sqcup \rho(D, \triangleright\gamma)$
$C \sqcup D$	$\triangleleft\gamma$	$\rho(C, \triangleleft\gamma) \sqcap \rho(D, \triangleleft\gamma)$
$\exists R.C$	$\triangleright\gamma$	$\exists\rho(R, \triangleright\gamma).\rho(C, \triangleright\gamma)$
$\exists R.C$	$\triangleleft\gamma$	$\forall\rho(R, \neg\triangleleft\gamma).\rho(C, \triangleleft\gamma)$
$\forall R.C$	$\geqslant \alpha$	$\sqcap_{\gamma\in\mathcal{TV}^+\vert\gamma\leqslant\alpha}(\forall\rho(R, \geqslant\gamma).\rho(C, \geqslant\gamma))\sqcap_{\gamma\in\mathcal{TV}\vert\gamma<\alpha}(\forall\rho(R, >\gamma).\rho(C, >\gamma))$
$\forall R.C$	$> \beta$	$\sqcap_{\gamma\in\mathcal{TV}^+\vert\gamma\leqslant\beta}(\forall\rho(R, \geqslant\gamma).\rho(C, \geqslant\gamma))\sqcap_{\gamma\in\mathcal{TV}\vert\gamma\leqslant\beta}(\forall\rho(R, >\gamma).\rho(C, >\gamma))$
$\forall R.C$	$\leqslant \beta$	$\sqcup_{\gamma\in\mathcal{TV}\vert\gamma\leqslant\beta}(\exists\rho(R, >\gamma).\rho(C, \leqslant\gamma))$
$\forall R.C$	$< \alpha$	$\sqcup_{\gamma\in\mathcal{TV}^+\vert\gamma\leqslant\alpha}(\exists\rho(R, \geqslant\gamma).\rho(C, <\gamma))$
$\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}$	$\bowtie\gamma$	$\{o_i \vert \alpha_i \bowtie \gamma, 1 \leqslant i \leqslant m\}$
$\geqslant m\ S.C$	$\triangleright\gamma$	$\geqslant m\rho(S, \triangleright\gamma).\rho(C, \triangleright\gamma)$
$\geqslant m\ S.C$	$\triangleleft\gamma$	$\leqslant m - 1\rho(S, \neg\triangleleft\gamma).\rho(C, \neg\triangleleft\gamma)$
$\leqslant n\ S.C$	$\triangleright\gamma$	$\leqslant n\rho(S, > 0).\rho(C, S, > 0)$
$\leqslant n\ S.C$	$\triangleleft\gamma$	$\geqslant n + 1\rho(S, > 0).\rho(C, S, > 0)$
$\exists S.\mathtt{Self}$	$\triangleright\gamma$	$\exists\rho(S, \triangleright\gamma).\mathtt{Self}$
$\exists S.\mathtt{Self}$	$\triangleleft\gamma$	$\neg\exists\rho(S, \neg\triangleleft\gamma).\mathtt{Self}$
$mod(C)$	$\bowtie\gamma$	See Section 5
$[C \geqslant \alpha]$	$\triangleright\gamma$	$\rho(C, \geqslant\alpha)$
$[C \geqslant \alpha]$	$\triangleleft\gamma$	$\rho(C, < \alpha)$

Table 4
Mapping of role expressions.

x	y	$\rho(x,y)$
R_A	$\triangleright\gamma$	$R_{A\triangleright\gamma}$
R_A	$\triangleleft\gamma$	$\neg R_{A\neg\triangleleft\gamma}$
U	$\triangleright\gamma$	U
U	$\triangleleft\gamma$	$\neg U$
R^-	$\bowtie\gamma$	$\rho(R,\bowtie\gamma)^-$
$mod(R)$	$\bowtie\gamma$	See Section 5
$[R \geqslant \alpha]$	$\triangleright\gamma$	$\rho(R,\geqslant\alpha)$
$[R \geqslant \alpha]$	$\triangleleft\gamma$	$\rho(R,<\alpha)$
$\neg R$	$\triangleright\gamma$	$\rho(R,\leqslant 0)$
$\neg R$	$\triangleleft\gamma$	$\rho(R,>0)$

Table 5
Reduction of the axioms.

Axiom	Reduction
$\kappa(\langle a : C \bowtie \gamma\rangle)$	$\{a : \rho(C,\bowtie\gamma)\}$
$\kappa(\langle (a,b) : R \bowtie \gamma\rangle)$	$\{(a,b) : \rho(R,\bowtie\gamma)\}$
$\kappa(\langle (a,b) : \neg R \bowtie \gamma\rangle)$	$\{(a,b) : \rho(\neg R,\bowtie\gamma)\}$
$\kappa(\langle a \neq b\rangle)$	$\{a \neq b\}$
$\kappa(\langle a = b\rangle)$	$\{a = b\}$
$\kappa(C \sqsubseteq D \geqslant \alpha)$	$\bigcup_{\gamma\in\mathcal{TV}^+\mid\gamma\leqslant\alpha}\{\rho(C,\geqslant\gamma)\sqsubseteq\rho(D,\geqslant\gamma)\}$ $\bigcup_{\gamma\in\mathcal{TV}\mid\gamma<\alpha}\{\rho(C,>\gamma)\sqsubseteq\rho(D,>\gamma)\}$
$\kappa(C \sqsubseteq D > \beta)$	$\kappa(C \sqsubseteq D \geqslant \beta) \cup \{\rho(C,>\beta)\sqsubseteq\rho(D,>\beta)\}$
$\kappa(\langle R_1 \ldots R_n \sqsubseteq R \geqslant \alpha\rangle)$	$\bigcup_{\gamma\in\mathcal{TV}^+\mid\gamma\leqslant\alpha}\{\rho(R_1,\geqslant\gamma)\ldots\rho(R_n,\geqslant\gamma)\sqsubseteq\rho(R,\geqslant\gamma)\}$ $\bigcup_{\gamma\in\mathcal{TV}\mid\gamma<\alpha}\{\rho(R_1,>\gamma)\ldots\rho(R_n,>\gamma)\sqsubseteq\rho(R,>\gamma)\}$
$\kappa(\langle R_1 \ldots R_n \sqsubseteq R > \beta\rangle)$	$\kappa(\langle R_1 \ldots R_n \sqsubseteq R \geqslant \beta\rangle)\cup\{\rho(R_1,>\beta)\ldots\rho(R_n,>\beta)\sqsubseteq\rho(R,>\beta)\}$
$\kappa(\mathtt{dis}(S_1,S_2))$	$\{\mathtt{dis}(\rho(S_1,>0),\rho(S_2,>0))\}$
$\kappa(\mathtt{ref}(R))$	$\{\mathtt{ref}(\rho(R,\geqslant 1))\}$
$\kappa(\mathtt{asy}(S))$	$\{\mathtt{asy}(\rho(S,>0)\}$

4.2. Mapping fuzzy concepts, roles and axioms

Fuzzy concept and role expressions are reduced using mapping ρ, as shown in Tables 3 and 4 respectively. Modifiers are discussed in Section 5. Given a fuzzy concept C, $\rho(C,\geqslant\alpha)$ is a crisp set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C,\bowtie\gamma)$ are similar. ρ is defined in a similar way for fuzzy roles and this equivalence also holds. It can be verified that $\rho(C,\bowtie\gamma)\equiv\neg\rho(C,\neg\bowtie\gamma)$.

Mapping ρ deserves some comments. Firstly, it is interesting to remark that $\rho(A,\leqslant\beta)=\neg A_{>\beta}$ is different from $\rho(\neg A,\geqslant\alpha)=\rho(A,\leqslant 0)=\neg A_{>0}$. Secondly, due to the restrictions in the definition of the fuzzy KB, some expressions cannot appear during the process:

- $\rho(A,\geqslant 0),\rho(A,>1),\rho(A,\leqslant 1),\rho(A,<0)$ cannot appear due to the existing restrictions on the degree of the axioms in the fuzzy KB. The same also holds for $\top$, $\bot$ and R_A.
- $\rho(R,\triangleleft\beta)$, $\rho(\neg R,\triangleleft\beta)$, $\rho([R\geqslant\alpha],\triangleleft\gamma)$ and $\rho(U,\triangleleft\beta)$ can only appear in a negated role assertion.

Axioms are reduced as in Table 5, where $\kappa(\tau)$ maps a fuzzy axiom τ in $\mathcal{GSROIQ}$ into a set of crisp axioms in $\mathcal{SROIQ}$. We note $\kappa(\mathcal{A})$ (resp. $\kappa(\mathcal{T})$, $\kappa(\mathcal{R})$) the union of the reductions of all the fuzzy axioms in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$).[3] Observe that $\kappa(\langle C\sqsubseteq D\geqslant 1\rangle)$ is equivalent to the reduction of a GCI under a semantics based on Zadeh's set inclusion proposed in [38], although this work introduces two unnecessary axioms $C_{\geqslant 0}\sqsubseteq D_{\geqslant 0}$ and $C_{>1}\sqsubseteq D_{>1}$.

Let us illustrate how the reduction of an axiom works by showing an example.

Example 4.1. Consider the GCI $\langle C\sqsubseteq D\geqslant\alpha\rangle$. If it is satisfied, $\inf_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\geqslant\alpha$. As this is true for the infimum, an arbitrary $x\in\Delta^{\mathcal{I}}$ must satisfy $C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\geqslant\alpha$. From the semantics of Gödel implication, this is true if $C^{\mathcal{I}}(x)\leqslant D^{\mathcal{I}}(x)$ or $D^{\mathcal{I}}(x)\geqslant\alpha$. Hence, for each $\gamma\in\mathcal{TV}^+$ such that $\gamma\leqslant\alpha$, $C^{\mathcal{I}}(x)\geqslant\gamma$ implies $D^{\mathcal{I}}(x)\geqslant\gamma$ (which is expressed as $\rho(C,\geqslant\gamma)\sqsubseteq\rho(D,\geqslant\gamma)$) and for each $\gamma\in\mathcal{TV}\mid\gamma<\alpha$, $C^{\mathcal{I}}(x)>\alpha$ implies $D^{\mathcal{I}}(x)>\alpha$ (which is expressed as $\rho(C,>\alpha)\sqsubseteq\rho(D,>\alpha)$).

Summing up, a fuzzy KB $\mathcal{K}=\langle\mathcal{A},\mathcal{T},\mathcal{R}\rangle$ is reduced into a KB $\mathcal{K}'=\langle\kappa(\mathcal{A}),T(\mathcal{TV})\cup\kappa(\mathcal{T}),R(\mathcal{TV})\cup\kappa(\mathcal{R})\rangle$. We highlight that the reduction preserves simplicity of the roles and regularity of the RIAs.

Now we will illustrate the whole procedure with an example.

[3] More precisely, the reduction of fuzzy GCIs and RIAs should be noted as $\kappa(\tau,\mathcal{TV})$, and the reduction of the fuzzy TBox and RBox as $\kappa(\mathcal{T},\mathcal{TV})$ and $\kappa(\mathcal{R},\mathcal{TV})$ respectively. For the sake of simplicity we omit $\mathcal{TV}$ since it is clear from the context.

Example 4.2. Consider a fuzzy KB $\mathcal{K}$ in a musical domain where we define $\mathcal{TV} = \{0, 0.3, 0.5, 0.7, 1\}$ and that represents the following knowledge:

- Radiohead is one of the favourite bands of Juan:

  $\langle(\texttt{juan}, \texttt{radiohead}) : \texttt{hasFavBand} \geqslant 1\rangle$
- Every Radiohead's record is not a live record (we introduce a degree to reflect the fact that there exist several non-official live records):

  $\langle\texttt{radiohead} : \forall\texttt{hasRecord}.(\neg\texttt{LiveRecord}) \geqslant 0.7\rangle$
- Fernando has at-most two favourite bands playing flamenco:

  $\langle\texttt{fernando} : \leqslant 2\texttt{hasFavBand}.\texttt{FlamencoBand} \geqslant 0.7\rangle$

Firstly, we create some new elements and some axioms preserving their semantics. $T(\mathcal{N}^{\mathcal{K}})$ contains the new axioms due to the new concepts:
$T(\mathcal{N}^{\mathcal{K}}) = \{\texttt{LiveRecord}_{\geqslant 1} \sqsubseteq \texttt{LiveRecord}_{>0.75}, \texttt{LiveRecord}_{>0.75} \sqsubseteq \texttt{LiveRecord}_{\geqslant 0.75}, \texttt{LiveRecord}_{\geqslant 0.75} \sqsubseteq \texttt{LiveRecord}_{>0.5}, \texttt{LiveRecord}_{>0.5} \sqsubseteq \texttt{LiveRecord}_{\geqslant 0.5}, \texttt{LiveRecord}_{\geqslant 0.5} \sqsubseteq \texttt{LiveRecord}_{>0.25}, \texttt{LiveRecord}_{>0.25} \sqsubseteq \texttt{LiveRecord}_{\geqslant 0.25}, \texttt{LiveRecord}_{\geqslant 0.25} \sqsubseteq \texttt{LiveRecord}_{>0}, \ldots\}$ (and analogously for FlamencoBand).

The case for the roles is similar, with $R(\mathcal{N}^{\mathcal{K}})$ containing the following set of axioms: $R(\mathcal{N}^{\mathcal{K}}) = \{\texttt{hasFavBand}_{\geqslant 1} \sqsubseteq \texttt{hasFavBand}_{>0.75}, \texttt{hasFavBand}_{>0.75} \sqsubseteq \texttt{hasFavBand}_{\geqslant 0.75}, \texttt{hasFavBand}_{\geqslant 0.75} \sqsubseteq \texttt{hasFavBand}_{>0.5}, \texttt{hasFavBand}_{>0.5} \sqsubseteq \texttt{hasFavBand}_{\geqslant 0.5}, \texttt{hasFavBand}_{\geqslant 0.5} \sqsubseteq \texttt{hasFavBand}_{>0.25}, \texttt{hasFavBand}_{>0.25} \sqsubseteq \texttt{hasFavBand}_{\geqslant 0.25}, \texttt{hasFavBand}_{\geqslant 0.25} \sqsubseteq \texttt{hasFavBand}_{>0}, \ldots\}$ (and analogously for the role hasRecord).

Finally, we map the three axioms in the ABox:

- $\kappa(\langle(\texttt{juan}, \texttt{radiohead}) : \texttt{hasFavBand} \geqslant 1\rangle) = (\texttt{juan}, \texttt{radiohead}) : \texttt{hasFavBand}_{\geqslant 1}$.
- $\kappa(\langle\texttt{radiohead} : \forall\texttt{hasRecord}.(\neg\texttt{LiveRecord}) \geqslant 0.7\rangle) =$

$$\begin{aligned}
&\texttt{radiohead} : \quad (\forall\texttt{hasRecord}_{>0}.\rho(\neg\texttt{LiveRecord}, > 0))\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.3}.\rho(\neg\texttt{LiveRecord}, \geqslant 0.3))\sqcap \\
&(\forall\texttt{hasRecord}_{>0.3}.\rho(\neg\texttt{LiveRecord}, > 0.3))\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.5}.\rho(\neg\texttt{LiveRecord}, \geqslant 0.5))\sqcap \\
&(\forall\texttt{hasRecord}_{>0.5}.\rho(\neg\texttt{LiveRecord}, > 0.5))\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.7}.\rho(\neg\texttt{LiveRecord}, \geqslant 0.7)) = \\
&\texttt{radiohead} : \quad (\forall\texttt{hasRecord}_{>0}.\texttt{LiveRecord}_{\leqslant 0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.3}.\texttt{LiveRecord}_{\leqslant 0})\sqcap \\
&(\forall\texttt{hasRecord}_{>0.3}.\texttt{LiveRecord}_{\leqslant 0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.5}.\texttt{LiveRecord}_{\leqslant 0})\sqcap \\
&(\forall\texttt{hasRecord}_{>0.5}.\texttt{LiveRecord}_{\leqslant 0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.7}.\texttt{LiveRecord}_{\leqslant 0}) = \\
&\texttt{radiohead} : \quad (\forall\texttt{hasRecord}_{>0}.\neg\texttt{LiveRecord}_{>0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.3}.\neg\texttt{LiveRecord}_{>0})\sqcap \\
&(\forall\texttt{hasRecord}_{>0.3}.\neg\texttt{LiveRecord}_{>0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.5}.\neg\texttt{LiveRecord}_{>0})\sqcap \\
&(\forall\texttt{hasRecord}_{>0.5}.\neg\texttt{LiveRecord}_{>0})\sqcap \\
&(\forall\texttt{hasRecord}_{\geqslant 0.7}.\neg\texttt{LiveRecord}_{>0})
\end{aligned}$$

- $\kappa(\langle\texttt{fernando} : \leqslant 2\texttt{hasFavBand}.\texttt{FlamencoBand} \geqslant 0.7\rangle) = \texttt{fernando} : \leqslant 2\texttt{hasFavBand}_{>0}.\texttt{FlamencoBand}_{>0}$.

Observe that the reduction of the second axiom can be simplified to

$$\texttt{radiohead} : \forall\texttt{hasRecord}_{>0}.\neg\texttt{LiveRecord}_{>0}$$

but in the general case the reduction of a fuzzy universal quantification is a conjunction of universal quantifications.

4.3. Correctness and complexity of the reduction

The following theorem shows the logic is decidable under Gödel semantics and that the reductions preserves reasoning.

Theorem 4.3. *The satisfiability problem in $\mathcal{GSROIQ}$ is decidable. Furthermore, a $\mathcal{GSROIQ}$ fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is satisfiable iff its crisp representation $\mathcal{K}' = \langle \kappa(\mathcal{A}), T(\mathcal{TV}) \cup \kappa(\mathcal{T}), R(\mathcal{TV}) \cup \kappa(\mathcal{R}) \rangle$ is satisfiable.*

Regarding the complexity, $|\mathcal{TV}|$ is bounded by $|\mathcal{K}| + 2$, and the size of the resulting KB is $\mathcal{O}(|\mathcal{TV}|^k)$,[4] where k is the maximal depth of the universal restriction concepts appearing which is inductively defined as follows:

- $depth(A) = depth(\top) = depth(\bot) = depth(\{\alpha_1/o_1, \ldots \alpha_m/o_m\}) = depth(\exists S.\ \texttt{Self}) = 1$,
- $depth(\exists R.C) = depth(\neg C) = depth(\geqslant n\ S.C) = depth(\leqslant n\ S.C) = depth(mod(C)) = depth(C \geqslant \alpha]) = depth(C)$,
- $depth(C \sqcap D) = depth(C \sqcup D) = max\{depth(C), depth(D)\}$,
- $depth(\forall R.C) = 1 + depth(C)$.

We recall that under Zadeh semantics, the size of the resulting KB is quadratic (or linear if we fix the number of degrees of truth). The increment of spatial complexity is due to the use of Gödel implication in universal restrictions. In this case it is not possible to infer the exact degrees of truth, but we need to guess them, building disjunctions or conjunctions over all possible combinations of the degrees of truth. However, in most of the cases universal restrictions of the form $(\forall R.C)$ can be approximated by using cut concepts and roles, replacing them by $(\forall [R \geqslant \alpha_1].[C \geqslant \alpha_2])$, meaning that every individual which is related through role R with degree (at least) α_1 must belong to C with (at least) degree α_2. Now the reduction is:

$$\begin{aligned} &\rho(\forall [R \geqslant \alpha_1].[C \geqslant \alpha_2], \rhd\gamma) = \forall \rho(R, \geqslant \alpha_1) \cdot \rho(C, \geqslant \alpha_2) \\ &\rho(\forall [R \geqslant \alpha_1] \cdot [C \geqslant \alpha_2], \lhd\gamma) = \exists \rho(R, \geqslant \alpha_1) \cdot \rho(C, < \alpha_2) \end{aligned} \tag{6}$$

Whenever this approximation is possible, the resulting KB is linear ($\mathcal{O}(|\mathcal{TV}|)$, as we are assuming a fixed finite set of degrees of truth $|\mathcal{TV}|$). From a practical point of view, in many applications it is sufficient to consider a small number of degrees, e.g. $\{0, 0.25, 0.5, 0.75, 1\}$, i.e. $\alpha \in \{0.25, 0.5, 0.75, 1\}$ and $\beta \in \{0, 0.25, 0.5, 0.75\}$.

Let $\kappa(\mathcal{K})$ denote the reduction of a fuzzy ontology $\mathcal{K}$. An interesting property of the procedure is that the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included. From an implementation point of view, this property allows to compute the reduction of the ontology off-line and update $\kappa(\mathcal{K})$ incrementally.

Theorem 4.4. *Let $\mathcal{K}$ be a $\mathcal{GSROIQ}$ fuzzy knowledge base involving a set of fuzzy atomic roles A and a set of atomic roles R, let $\mathcal{TV}$ be a fixed set of truth degrees including 0, 1, and the degrees in $\mathcal{K}$, and let τ be a $\mathcal{GSROIQ}$ axiom such that:*

1. *for every atomic concept A which appears in τ, $A \in A$,*
2. *for every atomic role R_A which appears in τ, $R_A \in R$,*
3. *if γ appears in τ, then $\gamma \in \mathcal{TV}$.*

Then, $\kappa(\mathcal{K} \cup \tau) = \kappa(\mathcal{K}) \cup \kappa(\tau)$.

The theorem assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often. These are reasonable assumptions because ontologies do not usually change once that their development has finished. Moreover, it has been shown that the set of the degrees which must be considered for any reasoning task is $\mathcal{TV}$. Regarding the computation of any greatest lower bound, we recall that U. Straccia has shown that, in the worst case, it requires to compute $log|\mathcal{TV}|$ satisfiability tests [35], which is another argument to fix the set of allowed degrees.

5. Crisp representation for modified fuzzy concepts and roles

In this section we will show how to extend our reduction in order to allow concept and role modifiers in the language. We will restrict ourselves to the triangular modifier and the linear modifier because they are suitable for a crisp representation of modified concepts and roles. Other fuzzy modifiers have been proposed in the literature (see Section 6 for a discussion).

The semantics of a *triangular modifier mTri* (Fig. 1a) is given by a function $f_{mTri}(x; t_1, t_2, t_3)$, where $t_1, t_2, t_3 \in [0, 1]$ and:

$$f_{mTri}(x; t_1, t_2, t_3) = \begin{cases} f_{left}(x; t_1, t_2, t_3) = t_1 + x(1 - t_1)/t_2 & x \in [0, t_2] \\ f_{right}(x; t_1, t_2, t_3) = 1 - (x - t_2)(1 - t_3)/(1 - t_2) & x \in [t_2, 1] \end{cases} \tag{7}$$

Note that $f_{mTri}(0) = t_1$, $f_{mTri}(t_2) = 1$ and $f_{mTri}(1) = t_3$.

The semantics of a *linear modifier mLin* (Fig. 1b) is given by a function $f_{mLin}(x; l)$, with $l \in [0, 1]$, $l_1 = \frac{l}{l+1}$ and $l_2 = \frac{1}{l+1}$, defined as follows:

$$f_{mLin}(x; l) = \begin{cases} (l_2/l_1)x & x \in [0, l_1] \\ 1 - (x - 1)(1 - l_2)/(1 - l_1) & x \in [l_1, 1] \end{cases} \tag{8}$$

Note that $f_{mLin}(0) = 0$, $f_{mLin}(l_1) = l_2$ and $f_{mLin}(1) = 1$.

[4] Fuzzy modified concept and roles do not introduce additional complexity as we will see in Section 5.

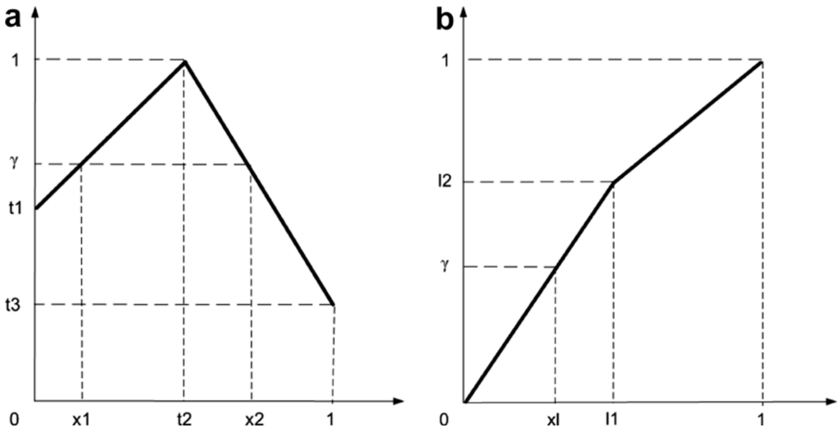

Fig. 1. (a) Triangular modifier; (b) linear modifier.

The first thing to be kept in mind is that it is no longer enough to consider the degrees in $\mathcal{TV}$. Consider a fuzzy KB with one assertion $\langle a : mod(C) \geqslant \gamma\rangle$, where the modifier *mod* is defined as in Fig. 1a. We can deduce that $\langle a : C \geqslant t_1\rangle$ and $\langle a : C \leqslant t_2\rangle$, so we should also consider the degree t_1, t_2 in $\mathcal{TV}$. But this is not enough, since we might have a concept of the form $mod(mod(mod(\ldots mod(C))\ldots))$.

Our solution to this problem is to restrict the membership function of every fuzzy modifier *mod* in such a way that $\forall\gamma \in \mathcal{TV}, f_{mod}(\gamma) \in \mathcal{TV}$.

Let $x_1 \in [0, b]$ and $x_2 \in [b, 1]$ be those numbers such that $f_{left}(x_1; t_1, t_2, t_3) = \gamma$ and $f_{right}(x_2; t_1, t_2, t_3) = \gamma$ respectively, for a triangular modifier *mTri*. Note that x_1 does not exist if $\gamma < t_1$, and that x_2 does not exist if $\gamma > t_2$. The reduction of modified concepts depend on the values of the parameters of the modifier, as Table 6 shows.

In the case of role modifiers, we only allow linear modifiers, because triangular modifiers would need to use role conjunction, role disjunction and expressions of the form $\rho(R, \triangleleft\gamma)$ outside the ABox, which are not part of crisp $\mathcal{SROIQ}$.

On the other hand, linear modifiers are reduced as:

$$\begin{aligned} \rho(mLin(C), \bowtie \gamma) &= \rho(C, \bowtie x_l) \\ \rho(mLin(R), \bowtie \gamma) &= \rho(R, \bowtie x_l) \end{aligned} \tag{9}$$

with x_l being that number such that $f_{mLin}(x_l; l) = \gamma$.

Example 5.1. Assume a fuzzy KB such that $\mathcal{TV} = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. Let us consider the reduction of the axiom $a : around(C) \geqslant 0.8$, where *around* is a triangular modifier defined as follows: $f_{around}(x; 0.6, 0.4, 0.4)$.

Table 6
Reduction of triangularly modified concepts.

Reduction of $\rho(mTri(C), \geqslant \gamma)$
if $(\alpha > t_1)$and$(\alpha > t_3)$ then $\rho(C, \geqslant x_1) \sqcap \rho(C, \leqslant x_2)$
if $(\alpha > t_1)$and$(\alpha \leqslant t_3)$ then $\rho(C, \geqslant x_1)$
if $(\alpha \leqslant t_1)$and$(\alpha > t_3)$ then $\rho(C, \leqslant x_2)$
if $(\alpha \leqslant t_1)$and$(\alpha \leqslant t_3)$ then $\top$

Reduction of $\rho(mTri(C), > \gamma)$
if $(\beta \geqslant t_1)$and$(\beta \geqslant t_3)$ then $\rho(C, > x_1) \sqcap \rho(C, < x_2)$
if $(\beta \geqslant t_1)$and$(\beta < t_3)$ then $\rho(C, > x_1)$
if $(\beta < t_1)$and$(\beta \geqslant t_3)$ then $\rho(C, < x_2)$
if $(\beta < t_1)$and$(\beta < t_3)$ then $\top$

Reduction of $\rho(mTri(C), \leqslant \gamma)$
if $(\beta \geqslant t_1)$and$(\beta \geqslant t_3)$ then $\rho(C, \leqslant x_1) \sqcup \rho(C, \geqslant x_2)$
if $(\beta \geqslant t_1)$and$(\beta < t_3)$ then $\rho(C, \leqslant x_1)$
if $(\beta < t_1)$and$(\beta \geqslant t_3)$ then $\rho(C, \geqslant x_2)$
if $(\beta < t_1)$and$(\beta < t_3)$ then $\bot$

Reduction of $\rho(mTri(C), < \gamma)$
if $(\alpha > t_1)$and$(\alpha > t_3)$ then $\rho(C, \geqslant x_1) \sqcup \rho(C, \leqslant x_2)$
if $(\alpha > t_1)$and$(\alpha \leqslant t_3)$ then $\rho(C, < x_1)$
if $(\alpha \leqslant t_1)$and$(\alpha > t_3)$ then $\rho(C, > x_2)$
if $(\alpha \leqslant t_1)$and$(\alpha \leqslant t_3)$ then $\bot$

Firstly, we have to verify that indeed $\forall \gamma \in \mathcal{TV}, f_{around}(\gamma) \in \mathcal{TV}$:

- $f_{around}(0) = 0.6 \in \mathcal{TV}$,
- $f_{around}(0.1) = 0.7 \in \mathcal{TV}$,
- $f_{around}(0.2) = 0.8 \in \mathcal{TV}$,
- $f_{around}(0.3) = 0.9 \in \mathcal{TV}$,
- $f_{around}(0.4) = 1 \in \mathcal{TV}$,
- $f_{around}(0.5) = 0.9 \in \mathcal{TV}$,
- $f_{around}(0.6) = 0.8 \in \mathcal{TV}$,
- $f_{around}(0.7) = 0.7 \in \mathcal{TV}$,
- $f_{around}(0.8) = 0.6 \in \mathcal{TV}$,
- $f_{around}(0.9) = 0.5 \in \mathcal{TV}$,
- $f_{around}(1) = 0.4 \in \mathcal{TV}$,

Now, x_1, x_2 are those points such that the modifier takes the value 0.8, so $x_1 = 0.2$ and $x_2 = 0.6$. Hence, the reduction of the axiom is $\kappa(\langle a : around(C) \geqslant 0.8 \rangle) = a : \rho(C, \geqslant x_1) \sqcap \rho(C, \leqslant x_2) = a : \rho(C, \geqslant 0.2) \sqcap \rho(C, \leqslant 0.6)$.

6. Related work

Since the first work of Yen [40], an important number of fuzzy extensions to DLs can be found in the literature [11]. In this section we will concentrate on the state of the art on families of fuzzy operators, modifiers and the representation of fuzzy DLs using crisp DLs.

Families of fuzzy operators. While most of the works restrict themselves to Zadeh family of fuzzy operators, a few other works consider Łukasiewicz logic. [30,41–43] propose a reasoning solution, which is based on a mixture of tableau rules and Mixed Integer Linear Programming (MILP) optimization problems. These works are implemented in the FUZZYDL reasoner [44].[5] Habiballa considers a fuzzy extension of $\mathcal{ALC}$ extended with role negation, top role and bottom role. He presents a novel reasoning algorithm based on resolution, as well as an implementation (GERDS) [45]. Another implementation based on resolution (YADLR) has been recently presented [46]. A proposal for a product t-norm-based fuzzy DL has also been presented [47], using Product logic but replacing Gödel negation with Łukasiewicz negation.

To the best of our knowledge, there are only two attempts towards a fuzzy DL based on Gödel logic. The first one is due to P. Hájek, who considered fuzzy $\mathcal{ALC}$ under arbitrary continuous t-norms and reported a reasoning algorithm based on a reduction to fuzzy propositional logic [36]. In this work, however, we reduce Gödel fuzzy DL to a crisp DL. The second one considers, in addition to minimum and maximum, Gödel implication, but only in the semantics of GCIs and RIAs [39]. In this work we use this implication also in universal quantification and qualified cardinality restrictions, and we use Gödel negation.

Fuzzy modifiers. The first work allowing concept modifiers is due to Tresp and Molitor, who considered manipulators (a special case of triangular membership functions) [48]. Hölldobler et al. have widely worked on this field. They proposed the use of exponential modifiers of the form $M(x) = x^{\beta}$. Initially, they only allowed to apply modifiers to atomic concepts [31], then they extended the work to complex concepts [49,50]. A later work considers linear modifiers which can be applied to concepts and (atomic) roles [51]. To the very best of our knowledge, it is the only previous proposal which allows to reason with role modifiers, but the expressivity of the logic ($\mathcal{ALC}$) is quite far from our work. The previous works present the problem that modifiers are not associative, which is solved in [52] (although they do not allow role modifiers here). As a minor comment, Singh et al. slightly changed the semantics of the modifiers in the context of an information retrieval problem application [53].

Straccia proposed a reasoning algorithm for fuzzy DLs based on a combination of a tableaux algorithm and a Mixed Integer Linear Programming (MILP) optimization problem [30]. This approach allows to use concept modifiers which are MILP representable. FUZZYDL reasoner is based on this idea, and it is the only current implementation allowing to use concept modifiers. In particular, it allows the use of modifiers defined in terms of linear hedges and triangular functions, as in this paper. Moreover, we have also allowed linear hedges to be applied to roles. Finally, Calegari et al. also suggested the use of role modifiers, but unfortunately they do not detail which membership function to use nor investigate how to reason with them [32,33].

Crisp representations for fuzzy DLs. The first effort in this direction is due to Straccia, who showed a reasoning preserving procedure for fuzzy $\mathcal{ALCH}$ [38]. A similar work from him considers fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice [54], therefore supporting quantitative reasoning (by using the interval $[0,1]$) and qualitative reasoning (e.g. by relying on a set"{`false`," "`likelyfalse`, `unknown`, `likelytrue`, `true`}"). Bobillo et al. widened the former work of Straccia to $\mathcal{SHOIN}$ and allowed fuzzy GCIs, but with a semantics given by KD implication [20]. Stoilos et al. extended this work and considered the reduction of an extension of fuzzy $\mathcal{SHOIN}$ with additional role axioms: general RIAs, reflexive, asymmetric and role disjointness axioms [28]. It is not a reduction of fuzzy $\mathcal{SROIQ}$ (not even $\mathcal{SROIN}$) because they do not show how to

[5] http://gaia.isti.cnr.it/ straccia/software/fuzzyDL/fuzzyDL.html.

reduce the universal role, qualified cardinality restrictions, local reflexivity concepts in expressions of the form $\rho(\exists S.\texttt{Self}, \triangleleft\gamma)$ nor negative role assertions. Moreover, GCIs and RIAs are forced to be either true or false. Bobillo et al. extended this work providing a crisp representation of full $\mathcal{SROIQ}$ with fuzzy GCIs and RIAs [39].

A different approach is due to Li et al., who propose a family of fuzzy Description Logics using α-cuts as atomic concept and roles [29]. The approach is slightly different to ours because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce an $\mathcal{ALCQ}$ ABox [55] and an $\mathcal{ALCH}$ concept [56] to their crisp versions. Nevertheless, both of these works assume an empty TBox. Finally, Dubois et al. combine possibilistic and fuzzy logics in the context of Description Logics (more concretely, in $\mathcal{ALCIN}(\circ)$) [57]. Interestingly, they also propose to represent every fuzzy set using two crisp sets (its support and its core) and comment the possibility of using more crisp sets, in order to have a more refined representation. Although for some applications this representation may be enough, there is a loss of information that does not occur in our approach.

All this previous work has been restricted to Zadeh family, with the exception of a reduction of $\mathcal{ALCHIO}$ under Łukasiewicz semantics [58]. This paper is the first contribution to provide a crisp representation for a fuzzy DL using Gödel semantics.

7. Conclusions and future work

This work has proved the decidability of the fuzzy DL $\mathcal{SROIQ}$ under Gödel semantics by proposing a reasoning preserving reduction to the crisp case. Assuming a finite set of degrees of truth including 0 and 1, the logic satisfies the witnessed model property. We have also shown how to represent additional constructors which are independent of the particular choice of the fuzzy operators: concept and role modifiers defined using triangular and linear functions, which are the only fuzzy concept modifiers which are currently being used in practical implementations (namely, the FUZZYDL reasoner). This is the first expressive fuzzy DL supporting reasoning with role modifiers.

The complexity of the resulting crisp KB is $\mathcal{O}(|\mathcal{TV}|^k)$, where k is the maximal depth of the universal restriction concepts and $|\mathcal{TV}|$ is the set of degrees of truth. Restricting the degrees of truth turns also to be essential in order to reuse the reduction of an ontology when adding new axioms (in this case it is only necessary to include the reduction of the new axioms) and to compute the greatest lower bound, since it needs to perform at-most $log|\mathcal{TV}|$ tests. Since we restrict the number of degrees of truth, if we approximate universal quantification concepts by using cut concepts and roles, then the resulting KBs are linear in size.

Providing a crisp representation for a fuzzy ontology allows to reuse current crisp ontology languages and reasoners, among other related resources. It supposes an important step towards the possibility of dealing with vagueness, offering several advantages:

- We can continue using standard languages with a lot of resources available, avoiding the need (and cost) of adapting them to the new fuzzy language.
- We may continue using existing crisp reasoners, which is important because nowadays there is no reasoner fully supporting a fuzzy extension of OWL 1.1, even under Zadeh semantics.

The main direction for future work is to implement the reduction and to perform an empirical evaluation. We plan to implement a tableau algorithm for the logic in order to compare the two approaches. It would also be interesting to study possible optimizations to the reduction process, similarly as in [39].

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Appendix A. Proofs

A.1. Proof of Proposition 3.1

1. $(\neg\exists R.C)^{\mathcal{I}}(x) = \ominus(\sup_{y\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y)\otimes C^{\mathcal{I}}(y)\}$. There are two possibilities.
   - $(\neg\exists R.C)^{\mathcal{I}}(x) = 1$ if the supremum is 0, that is, $\forall y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y)\otimes C^{\mathcal{I}}(y) = 0$, which is true if $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $C^{\mathcal{I}}(y) = 0$ holds. In other words, the value is 1 if there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$.
   - $(\neg\exists R.C)^{\mathcal{I}}(x) = 0$ if the supremum is greater than 0, that is, $\exists y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) > 0$, which is true if $\exists y \in \Delta^{\mathcal{I}}$, with $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. In other words, the value is 0 if there exists some element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$.

Now, consider $(\forall R.(\neg C))^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\}$. Firstly, assume that there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. Then, $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $C^{\mathcal{I}}(y) = 0$ hold, which is equivalent to say that $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $(\neg C)^{\mathcal{I}}(y) = 1$ hold.

- If $R^{\mathcal{I}}(x,y) = 0$, then $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y) = 0 \Rightarrow (\neg C)^{\mathcal{I}}(y) = 1$.
- If $(\neg C)^{\mathcal{I}}(y) = 1$, then $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y) = R^{\mathcal{I}}(x,y) \Rightarrow 1 = 1$.

In any case, we end up with $\inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\} = 1$. Finally, assume that there exists some element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. Then, $\exists y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$ holds. Hence, $\exists y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x,y) > 0$ and $(\neg C)^{\mathcal{I}}(y) = 0$ holds, and hence it satisfies $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\} = 0$. So, $\inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\} = 0$. Summing up, in any case (either there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$, or there exists such an element), $\neg \exists R.C \equiv \forall R.(\neg C)$.

2. $(\leqslant n\ S.C)^{\mathcal{I}}(x) = \inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})]$. Note that $(\oplus_{j<k}\{y_j = y_k\})$ can be either 0 or 1, so the result of the Gödel implication is either 0 or 1 and hence $(\leqslant n\ S.C)$ is actually a crisp concept.
   - Assume that $\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 0$. Then, there exist $y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}$ such that $[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 0$. This is true if there exist $n+1$ mutually different elements y_i such that $(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) > 0$, that is, $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i)\}) > 0$.
   - Assume that $\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 1$. Then, $\forall y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}, [(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 1$. This is true in two cases:
     – $(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) = 0$, so there exist some y_i such that $S^{\mathcal{I}}(x,y_i) = 0$ or $C^{\mathcal{I}}(y_i) = 0$ hold.
     – $\oplus_{j<k}\{y_j = y_k\} = 0$ holds.
     This means that there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$.

Now, consider $(\neg(\geqslant n+1\ S.C))^{\mathcal{I}}(x) = \ominus(\sup_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\}))$. Firstly, assume that there exist $n+1$ mutually different elements y_i such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i)\}) > 0$. Then, $\sup_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\}) > 0$, so $\ominus(\sup_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\})) = 0$. Now, assume that there exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$. Then, $\sup_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\}) = 0$, so $\ominus(\sup_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \otimes (\otimes_{j<k}\{y_j \neq y_k\})) = 1$. Summing up, in any case (either there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$ or $C^{\mathcal{I}}(y) > 0$, or there do exist such elements), $(\leqslant n\ S.C) \equiv \neg(\geqslant n+1\ S.C)$. □

A.2. Proof of Proposition 3.3

(*i*) $\langle a : C \triangleright \alpha\rangle$ implies $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \alpha$. $\langle C \sqsubseteq D \triangleright \beta\rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \beta$. Since this is true for the infimum, it is also true for $a^{\mathcal{I}}$, so $C^{\mathcal{I}}(a^{\mathcal{I}}) \Rightarrow D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \beta$. But from $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \alpha$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \Rightarrow D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \beta$, using modus ponens with Gödel implication (see Section 2.2), it follows that $D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \min\{\alpha,\beta\}$. Hence, $\langle a : D \triangleright \min\{\alpha,\beta\}\rangle$ holds.

(*ii*) $\langle (a,b) : R \triangleright \alpha\rangle$ implies $R^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}) \triangleright \alpha$. $\langle R \sqsubseteq R' \triangleright \beta\rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow R'^{\mathcal{I}}(x,y) \triangleright \beta$. Since this is true for the infimum, in particular $R^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}) \Rightarrow R'^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}) \triangleright \beta$. Similarly as in the previous case, using modus ponens with Gödel implication, it follows that $R'^{\mathcal{I}}(a^{\mathcal{I}},b^{\mathcal{I}}) \triangleright \min\{\alpha,\beta\}$. Hence, $\langle (a,b) : R' \triangleright \min\{\alpha,\beta\}\rangle$ holds.

(*iii*) $\langle C \sqsubseteq D \triangleright \alpha\rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \alpha$, and $\langle D \sqsubseteq E \triangleright \beta\rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} D^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$. Now, for an individual x there are three possibilities:

1. $C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x) \leqslant E^{\mathcal{I}}(x)$. It follows that $C^{\mathcal{I}}(x) \leqslant E^{\mathcal{I}}(x)$ and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) = 1 \triangleright \min\{\alpha,\beta\}$.
2. $C^{\mathcal{I}}(x) > D^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x) \leqslant E^{\mathcal{I}}(x)$. From $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \alpha$ it follows that $D^{\mathcal{I}}(x) \triangleright \alpha$. Since $E^{\mathcal{I}}(x) \geqslant D^{\mathcal{I}}(x)$, it then $E^{\mathcal{I}}(x) \triangleright \alpha$. Since the result of $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x)$ under Gödel implication is either 1 or $E^{\mathcal{I}}(x)$, $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \alpha$, and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \min\{\alpha,\beta\}$.
3. $D^{\mathcal{I}}(x) > E^{\mathcal{I}}(x)$. From $D^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$ it follows that $E^{\mathcal{I}}(x) \triangleright \beta$. Since the result of $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x)$ under Gödel implication is either 1 or $E^{\mathcal{I}}(x)$, $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$, and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \min\{\alpha,\beta\}$.

In summary, $\forall x \in \Delta^{\mathcal{I}}$ we can always conclude that $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \min\{\alpha,\beta\}$, so $\langle C \Rightarrow E \triangleright \min\{\alpha,\beta\}\rangle$ holds.

(*iv*) $\langle R \sqsubseteq R' \triangleright \alpha\rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow R'^{\mathcal{I}}(x,y) \triangleright \alpha$, and $\langle R' \sqsubseteq R'' \triangleright \beta\rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R'^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \beta$. Now, for a pair of individuals x,y there are three possibilities:

1. $R^{\mathcal{I}}(x,y) \leqslant R'^{\mathcal{I}}(x,y)$ and $R'^{\mathcal{I}}(x,y) \leqslant R''^{\mathcal{I}}(x,y)$. It follows that $R^{\mathcal{I}}(x,y) \leqslant R''^{\mathcal{I}}(x,y)$ and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) = 1 \triangleright \min\{\alpha,\beta\}$.
2. $R^{\mathcal{I}}(x,y) > R'^{\mathcal{I}}(x,y)$ and $R'^{\mathcal{I}}(x,y) \leqslant R''^{\mathcal{I}}(x,y)$. From $R^{\mathcal{I}}(x,y) \Rightarrow R'^{\mathcal{I}}(x,y) \triangleright \alpha$ it follows that $R'^{\mathcal{I}}(x,y) \triangleright \alpha$. Since $R''^{\mathcal{I}}(x,y) \geqslant R'^{\mathcal{I}}(x,y)$, then $R''^{\mathcal{I}}(x,y) \triangleright \alpha$. Since the result of $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y)$ under Gödel implication is either 1 or $R''^{\mathcal{I}}(x,y)$, $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \alpha$, and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \min\{\alpha,\beta\}$.
3. $R'^{\mathcal{I}}(x,y) > R''^{\mathcal{I}}(x,y)$. From $R'^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \beta$ it follows that $R''^{\mathcal{I}}(x,y) \triangleright \beta$. Since the result of $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y)$ under Gödel implication is either 1 or $R''^{\mathcal{I}}(x,y)$, $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \beta$, and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \min\{\alpha,\beta\}$.

Hence, in every case $\forall x,y \in \Delta^{\mathcal{I}}$ we can conclude that $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \triangleright \min\{\alpha,\beta\}$, so $\langle R \Rightarrow R'' \triangleright \min\{\alpha,\beta\}\rangle$ holds.

A.3. Proof of Proposition 3.2

On the one hand, $\langle a : C \geqslant \alpha\rangle$ implies $C^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant \alpha$. On the other hand, from $\langle\{\alpha/a\} \sqsubseteq C \geqslant 1\rangle$ and under an R-implication, we can deduce that, for every individual x of the domain, $(\{\alpha/a\})^{\mathcal{I}}(x) \leqslant C^{\mathcal{I}}(x)$. In particular, for $a^{\mathcal{I}}$ we have that $C^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant (\{\alpha/a\})^{\mathcal{I}}(a^{\mathcal{I}}) = \alpha$.

A.4. Proof of Proposition 3.4

- On the one hand, $\mathtt{irr}(S)$ implies that $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$. On the other hand, $\langle \top \sqsubseteq \neg\exists S.\mathtt{Self} \geqslant 1 \rangle$ implies that $\inf_{x\in\Delta^{\mathcal{I}}} (\top)^{\mathcal{I}}(x) \Rightarrow (\neg\exists S.\mathtt{Self})^{\mathcal{I}}(x) \geqslant 1$. Under an R-implication, it follows that $(\top)^{\mathcal{I}}(x) = 1 \leqslant (\neg\exists S.\mathtt{Self})^{\mathcal{I}}(x)$. Due to the standard properties of negation functions, $(\neg\exists S.\mathtt{Self})^{\mathcal{I}}(x) \geqslant 1$ implies that $(\exists S.\mathtt{Self})^{\mathcal{I}}(x) = 0$. Hence, $\forall x \in \Delta^{\mathcal{I}}, (\exists S.\mathtt{Self})^{\mathcal{I}}(x) = S^{\mathcal{I}}(x,x) = 0$.
- On the one hand, $\mathtt{trans}(R)$ implies that $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geqslant \sup_{z\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$. On the other hand, $\langle RR \sqsubseteq R \geqslant 1 \rangle$ implies that, $\inf_{x,y\in\Delta^{\mathcal{I}}} \sup_{z\in\Delta^{\mathcal{I}}} (R_1^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)) \Rightarrow R^{\mathcal{I}}(x,y) \geqslant 1$. Under an R-implication, it follows that $\forall_{x,y\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \geqslant \sup_{z\in\Delta^{\mathcal{I}}} R_1^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$.
- On the one hand, $\mathtt{sym}(R)$ implies that $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. On the other hand, $\langle R \sqsubseteq R^- \geqslant 1 \rangle$ implies that, $\inf_{x,y\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow (R^-)^{\mathcal{I}}(x,y) \geqslant 1$. For any pair of individuals (x,y) it follows that $R^{\mathcal{I}}(x,y) \Rightarrow (R^-)^{\mathcal{I}}(x,y) \geqslant 1$. Under an R-implication, $R^{\mathcal{I}}(x,y) \leqslant (R^-)^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. But if we consider the pair (y,x), it follows than $R^{\mathcal{I}}(y,x) \leqslant (R^-)^{\mathcal{I}}(y,x) = R^{\mathcal{I}}(x,y)$. Hence, $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$.

A.5. Proof of Theorem 4.3

We will show the proof for the only-if direction. From $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ satisfying every axiom in $\mathcal{K}$. Now, it is possible to build a (crisp) interpretation $\mathcal{I}_{\mathcal{C}} = \{\Delta^{\mathcal{I}_{\mathcal{C}}}, \cdot^{\mathcal{I}_{\mathcal{C}}}\}$ as:

- $\Delta^{\mathcal{I}_{\mathcal{C}}} = \Delta^{\mathcal{I}}$.
- $a^{\mathcal{I}_{\mathcal{C}}} = a^{\mathcal{I}}$, for every individual a.
- $A_{\geqslant\alpha}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}} | A^{\mathcal{I}}(x) \geqslant \alpha\}$, for each $A \in \mathcal{K}$ and $\alpha \in \mathcal{TV} \setminus \{0\}$.
- $A_{>\beta}^{\mathcal{I}_{\mathcal{C}}} = \{x \in \Delta^{\mathcal{I}} | A^{\mathcal{I}}(x) > \beta\}$, for each $A \in \mathcal{K}$, $\beta \in \mathcal{TV} \setminus \{1\}$.
- $R_{A\geqslant\alpha}^{\mathcal{I}_{\mathcal{C}}} = \{x,y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | R_A^{\mathcal{I}}(x,y) \geqslant \alpha\}$, for each $R_A \in \mathcal{K}$, $\alpha \in \mathcal{TV} \setminus \{0\}$.
- $R_{A>\beta}^{\mathcal{I}_{\mathcal{C}}} = \{x,y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | R_A^{\mathcal{I}}(x,y) > \beta\}$, for each $R_A \in \mathcal{K}$, $\beta \in \mathcal{TV} \setminus \{1\}$.

Now, we will show that $\mathcal{I}_{\mathcal{C}}$ satisfies every axiom in the crisp representation $\kappa(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{TV}) \cup \kappa(\mathcal{T}), R(\mathcal{TV}) \cup \kappa(\mathcal{R}) \rangle$. For every axiom $\tau \in \mathcal{K}$, there are several cases:

1. τ is an inequality assertion. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_{\mathcal{C}}} \neq b^{\mathcal{I}_{\mathcal{C}}}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a \neq b \rangle \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a \neq b \rangle)$.
2. τ is an equality assertion. Assume that $\mathcal{I} \models \langle a = b \rangle$. Then, $a^{\mathcal{I}} = b^{\mathcal{I}}$. By definition of $\mathcal{I}_{\mathcal{C}}$, $a^{\mathcal{I}_{\mathcal{C}}} = b^{\mathcal{I}_{\mathcal{C}}}$, so $\mathcal{I}_{\mathcal{C}} \models \langle a = b \rangle \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a = b \rangle)$.
3. τ is a role assertion. Assume that $\mathcal{I} \models \langle (a,b) : R \bowtie \gamma \rangle$. We show, by induction on the structure of roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : R \bowtie \gamma \rangle)$.
   - *atomic role*. Assume that $\mathcal{I} \models \langle (a,b) : R_A \triangleright \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in R_{A\triangleright\gamma}^{\mathcal{I}_{\mathcal{C}}}$. By definition of ρ, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R_A, \triangleright\gamma))^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R_A, \triangleright\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : R_A \triangleright \gamma \rangle)$. Now assume that $\mathcal{I} \models \langle (a,b) : R_A \triangleleft \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleleft \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \notin (R_{A\neg\triangleright\gamma})^{\mathcal{I}_{\mathcal{C}}}$ and hence $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\neg R_{A\neg\triangleright\gamma})^{\mathcal{I}_{\mathcal{C}}}$. By definition of ρ, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R_A, \triangleleft\gamma))^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R_A, \triangleleft\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : R_A \triangleleft \gamma \rangle)$.
   - *negated role*. Assume that $\mathcal{I} \models \langle (a,b) : \neg R \geqslant \alpha \rangle$. Then, $\ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geqslant \alpha$. Since the result of Gödel negation is either 0 or 1, and given that $\alpha > 0$, it follows that $\ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$ and hence $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 0$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \notin \rho(R, > 0)^{\mathcal{I}_{\mathcal{C}}} \iff (a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in \rho(R, \leqslant 0)^{\mathcal{I}_{\mathcal{C}}} \iff (a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in \rho(\neg R, \geqslant \alpha)^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(\neg R, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : \neg R \geqslant \alpha \rangle)$. The other cases are similar.
   - *inverse role*. Assume that $\mathcal{I} \models \langle (a,b) : R^- \bowtie \gamma \rangle$. Then, $R^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}_{\mathcal{C}}}, a^{\mathcal{I}_{\mathcal{C}}}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}_{\mathcal{C}}} \iff (a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(R, \bowtie \gamma)^{\mathcal{I}_{\mathcal{C}}})^- \iff \mathcal{I}_{\mathcal{C}} \models (a,b) \in \rho(R, \bowtie \gamma)^- \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : R^- \bowtie \gamma \rangle)$.
   - *universal role*. Assume that $\mathcal{I} \models \langle (a,b) : U \triangleright \gamma \rangle$. Then, $U^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1 \geqslant \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in \Delta^{\mathcal{I}_{\mathcal{C}}} \times \Delta^{\mathcal{I}_{\mathcal{C}}}$ and consequently $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in U^{\mathcal{I}_{\mathcal{C}}} \iff (a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in (\rho(U, \triangleright\gamma))^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(U, \triangleright\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : U \triangleright \gamma \rangle)$. The case $\mathcal{I} \models \langle (a,b) : U \triangleleft \gamma \rangle$ is similar.
   - *modified role*. Assume that $\mathcal{I} \models \langle (a,b) : mLin(R) \triangleright \gamma \rangle$ for a linear modifier $mLin$ such that $f_{mLin}(x; l)$. Then, it follows that $f_{mLin}(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}); l) \triangleright \gamma$. Let $x_l \in [0,1]$ be such that $f_{mLin}(x_l; l) = \gamma$. Then, it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright x_l$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}}) \in \rho(R, \triangleright x_l)^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R, \triangleright x_l) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : mLin(R) \triangleright \gamma \rangle)$. The case $\mathcal{I} \models \langle (a,b) : mLin(R) \triangleleft \gamma \rangle$ is similar, but now it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleleft x_l$ so we end up with $\mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R, \triangleleft x_l) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : mLin(R) \triangleleft \gamma \rangle)$.
   - *cut role*. Assume that $\mathcal{I} \models \langle (a,b) : [R \geqslant \alpha] \triangleright \gamma \rangle$. Then, it follows that $([R \geqslant \alpha])^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$, which is the case if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geqslant \alpha$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}}, b^{\mathcal{I}_{\mathcal{C}}}) \in \rho(R, \geqslant \alpha)^{\mathcal{I}_{\mathcal{C}}} \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho([R \geqslant \alpha], \triangleright\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa((a,b) : [R \geqslant \alpha] \triangleright \gamma)$. The case $\mathcal{I} \models \langle (a,b) : [R \geqslant \alpha] \triangleleft \gamma \rangle$ is similar.

4. τ is a concept assertion. Assume $\mathcal{I} \models \langle a : C \bowtie \gamma\rangle$. We show, by induction on the structure of concepts and roles, that $\mathcal{I}_\mathcal{C} \models \kappa(\langle a : C \bowtie \gamma\rangle)$.
   - *atomic concept*. Assume that $\mathcal{I} \models \langle a : A \triangleright \gamma\rangle$. Then, $A^\mathcal{I}(a^\mathcal{I}) \triangleright \gamma$. By definition of $\mathcal{I}_\mathcal{C}$, it follows that $a^{\mathcal{I}_\mathcal{C}} : A^{\mathcal{I}_\mathcal{C}}_{\triangleright\gamma}$. By definition of ρ, $a^{\mathcal{I}_\mathcal{C}} \in (\rho(A, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(A, \triangleright\gamma) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : A \triangleright \gamma\rangle)$. Now assume that $\mathcal{I} \models \langle a : A \triangleleft \gamma\rangle$. Then, $A^\mathcal{I}(a^\mathcal{I}) \triangleleft \gamma$. By definition of $\mathcal{I}_\mathcal{C}$, it follows that $a^{\mathcal{I}_\mathcal{C}} \notin A^{\mathcal{I}_\mathcal{C}}_{\neg\triangleright\gamma} \iff a^{\mathcal{I}_\mathcal{C}} \in \neg A^{\mathcal{I}_\mathcal{C}}_{\neg\triangleright\gamma} \iff a^{\mathcal{I}_\mathcal{C}} \in (\rho(A, \triangleleft\gamma))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(A, \triangleleft\gamma) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : A \triangleleft \gamma\rangle)$.
   - *top concept*. Assume that $\mathcal{I} \models \langle a : \top \triangleright \gamma\rangle$. Then, $\top^\mathcal{I}(a^\mathcal{I}) \triangleright \gamma$. By definition of $\mathcal{I}_\mathcal{C}$, it follows that $a^{\mathcal{I}_\mathcal{C}} \in \Delta^{\mathcal{I}_\mathcal{C}} = \top$. By definition of ρ,, $a^{\mathcal{I}_\mathcal{C}} \in (\rho(\top, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(\top, \triangleright\gamma) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \top \triangleright \gamma\rangle)$. The case $\mathcal{I} \models \langle a : \top \triangleleft \gamma\rangle$ is not possible. If $\mathcal{I} \models \langle a : \top \leqslant \beta\rangle$ we have that $1 \leqslant \beta$, which is contradiction with the restriction $\beta \in [0, 1)$. If $\mathcal{I} \models \langle a : \top < \alpha\rangle$ we have that $1 < \alpha$, which is contradiction with the restriction $\alpha \in (0, 1]$.
   - *bottom concept*. This case is similar to the previous one.
   - *concept negation*. Assume that $\mathcal{I} \models \langle a : \neg C \geqslant \alpha\rangle$. Then, $\ominus C^\mathcal{I}(a^\mathcal{I}) \geqslant \alpha$. Since $\alpha > 0$, it follows that $\ominus C^\mathcal{I}(a^\mathcal{I}) = 1$ and hence $C^\mathcal{I}(a^\mathcal{I}) = 0$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \notin \rho(C, > 0)^{\mathcal{I}_\mathcal{C}} \iff a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \leqslant 0)^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(C, \leqslant 0) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \neg C \geqslant \alpha\rangle)$. The case $> \beta$ is similar. In the case $\mathcal{I} \models \langle a : \neg C \leqslant \beta\rangle$ it follows that $\ominus C^\mathcal{I}(a^\mathcal{I}) \leqslant \beta$. Since $\beta < 1$, it follows that $\ominus C^\mathcal{I}(a^\mathcal{I}) = 0$ and hence $C^\mathcal{I}(a^\mathcal{I}) > 0$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, > 0)^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(C, > 0) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \neg C \leqslant \beta\rangle)$. The case $< \alpha$ is similar.
   - *concept conjunction*. Assume that $\mathcal{I} \models \langle a : C \sqcap D \triangleright \gamma\rangle$. Then, $\min\{C^\mathcal{I}(a^\mathcal{I}), D^\mathcal{I}(a^\mathcal{I})\} \triangleright \gamma$, so it follows that $C^\mathcal{I}(a^\mathcal{I}) \triangleright \gamma$ and $D^\mathcal{I}(a^\mathcal{I}) \triangleright \gamma$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \triangleright\gamma)^{\mathcal{I}_\mathcal{C}}$ and $a^{\mathcal{I}_\mathcal{C}} \in \rho(D, \triangleright\gamma)^{\mathcal{I}_\mathcal{C}}$. By definition of ρ, $a^{\mathcal{I}_\mathcal{C}} \in (\rho(C, \triangleright\gamma) \sqcap \rho(D, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}} \iff a^{\mathcal{I}_\mathcal{C}} \in (\rho(C \sqcap D, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(C \sqcap D, \triangleright\gamma) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : C \sqcap D \triangleright \gamma\rangle)$. In the case $\mathcal{I} \models \langle a : C \sqcap D \triangleleft \gamma\rangle$, it follows that $C^\mathcal{I}(a^\mathcal{I}) \triangleleft \gamma$ or $D^\mathcal{I}(a^\mathcal{I}) \triangleleft \gamma$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \triangleleft\gamma)^{\mathcal{I}_\mathcal{C}}$ or $a^{\mathcal{I}_\mathcal{C}} \in \rho(D, \triangleleft\gamma)^{\mathcal{I}_\mathcal{C}}$. In this case, we end up with $\mathcal{I}_\mathcal{C} \models \kappa(\langle a : C \sqcap D \triangleleft \gamma\rangle)$.
   - *concept disjunction*. This case is similar to concept conjunction.
   - *universal quantification*. Assume that $\mathcal{I} \models \langle a : \forall R.C \geqslant \alpha\rangle$. Then, $\inf_{b \in \Delta^\mathcal{I}}\{R^\mathcal{I}(a^\mathcal{I}, b) \Rightarrow C^\mathcal{I}(b)\} \geqslant \alpha$. Since this is true for the infimum, an arbitrary individual $b \in \Delta^\mathcal{I}$ must satisfy that $R^\mathcal{I}(a^\mathcal{I}, b) \Rightarrow C^\mathcal{I}(b) \geqslant \alpha$ and hence one of the following conditions holds (*i*) $R^\mathcal{I}(a^\mathcal{I}, b) \leqslant C^\mathcal{I}(b)$ (which makes the Gödel implication equal to $1 \geqslant \alpha$), or (*ii*) $C^\mathcal{I}(b) \geqslant \alpha$ (which makes the Gödel implication take a value $\geqslant \alpha$). The former condition is equivalent to $R^\mathcal{I}(a^\mathcal{I}, b) \geqslant \gamma$ implies $C^\mathcal{I}(b) \geqslant \gamma$ for every $\gamma \in \mathcal{TV}$.[6] The latter condition allows us to restrict to those $\gamma \in \mathcal{TV}$ such that $\gamma \leqslant \alpha$. By induction hypothesis, it follows that $(a^{\mathcal{I}_\mathcal{C}}, b) \in (\rho(R, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}}$ implies $b^{\mathcal{I}_\mathcal{C}} \in (\rho(C, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}}$ or $b^{\mathcal{I}_\mathcal{C}} \in (\rho(C, \geqslant \alpha))^{\mathcal{I}_\mathcal{C}}$ for an arbitrary $b \in \Delta^{\mathcal{I}_\mathcal{C}}$. Consequently, it follows that $\mathcal{I}_\mathcal{C} \models a : \sqcap_{\gamma \in \mathcal{TV}\setminus\{0\} | \gamma \leqslant \alpha}(\forall\rho(R, \geqslant \gamma).\rho(C, \geqslant \gamma))\sqcap_{\gamma \in \mathcal{TV} | \gamma < \alpha}(\forall\rho(R, > \gamma).\rho(C, > \gamma)) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \forall R.C \geqslant \alpha\rangle)$. The case for $> \beta$ is quite similar.Now assume that $\mathcal{I} \models \langle a : \forall R.C \leqslant \beta\rangle$. Then, $\inf_{b \in \Delta^\mathcal{I}}\{R^\mathcal{I}(a^\mathcal{I}, b) \Rightarrow C^\mathcal{I}(b)\} \leqslant \beta$. Due to the witnessed model property, there is an individual $b \in \Delta^\mathcal{I}$ satisfying that $R^\mathcal{I}(a^\mathcal{I}, b) \Rightarrow C^\mathcal{I}(b)\} \leqslant \beta$. Since $\beta < 1$, it follows that (*i*) $R^\mathcal{I}(a^\mathcal{I}, b) > C^\mathcal{I}(b)$, and (*ii*) $C^\mathcal{I}(b) \leqslant \beta$. In this case we end up with $\mathcal{I}_\mathcal{C} \models a : \bigsqcup_{\gamma \in \mathcal{TV} | \gamma \leqslant \beta}(\exists\rho(R, > \gamma).\rho(C, \leqslant \gamma))$ and hence $\mathcal{I}_\mathcal{C} \models \kappa(\langle a : \forall R.C \leqslant \beta\rangle)$. The case for $< \alpha$ is quite similar.
   - *existential quantification*. Assume that $\mathcal{I} \models \langle a : \exists R.C \triangleright \gamma\rangle$. Then, $\sup_{b \in \Delta^\mathcal{I}} \min\{R^\mathcal{I}(a^\mathcal{I}, b), C^\mathcal{I}(b)\} \triangleright \gamma$. Due to the witnessed model property, there exists an individual b satisfying $\min\{R^\mathcal{I}(a^\mathcal{I}, b), C^\mathcal{I}(b)\} \triangleright\gamma$, so $R^\mathcal{I}(a^\mathcal{I}, b) \triangleright \gamma$ and $C^\mathcal{I}(b) \triangleright \gamma$. By induction hypothesis, $(a^{\mathcal{I}_\mathcal{C}}, b) \in (\rho(R, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}}$ and $b \in (\rho(C, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}}$ for some individual $b \in \Delta^{\mathcal{I}_\mathcal{C}}$, which is equivalent to $a^{\mathcal{I}_\mathcal{C}} \in (\exists\rho(R, \triangleright\gamma).\rho(C, \triangleright\gamma))^{\mathcal{I}_\mathcal{C}}$. By definition of ρ, $a^{\mathcal{I}_\mathcal{C}} \in (\rho(\exists R.C \triangleright \gamma))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(\exists R.C \triangleright \gamma) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \exists R.C, \triangleright\gamma\rangle)$. Now, assume that $\mathcal{I} \models \langle a : \exists R.C \leqslant \beta\rangle$. Then, $\sup_{b \in \Delta^\mathcal{I}} \min\{R^\mathcal{I}(a^\mathcal{I}, b), C^\mathcal{I}(b)\} \leqslant \beta$. Since this is true for the supremum, an arbitrary individual $b \in \Delta^\mathcal{I}$ must satisfy $R^\mathcal{I}(a^\mathcal{I}, b) \leqslant \beta$ or $C^\mathcal{I}(b) \leqslant \beta$. By induction hypothesis, $(a^{\mathcal{I}_\mathcal{C}}, b) \in (\rho(R, \leqslant \beta))^{\mathcal{I}_\mathcal{C}}$ or $b \in (\rho(C, \leqslant \beta))^{\mathcal{I}_\mathcal{C}}$ for some individual $b \in \Delta^{\mathcal{I}_\mathcal{C}}$, which is equivalent to $a^{\mathcal{I}_\mathcal{C}} \in (\forall\rho(R, > \beta).\rho(C, \leqslant \beta))^{\mathcal{I}_\mathcal{C}} \iff a^{\mathcal{I}_\mathcal{C}} \in (\forall\rho(R, \neg \leqslant \beta).\rho(C, \leqslant \beta))^{\mathcal{I}_\mathcal{C}}$. By definition of ρ, $a^{\mathcal{I}_\mathcal{C}} \in (\rho(\exists R.C \leqslant \beta))^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(\exists R.C \leqslant \beta) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \exists R.C, \leqslant \beta\rangle)$. The case $< \alpha$ is similar.
   - *fuzzy nominals*. Assume that $\mathcal{I} \models \langle a : \{\alpha_1/o_1, \ldots, \alpha_n/o_n\} \triangleright \gamma\rangle$. Let $o_{i1}, \ldots, o_{ik}$ be such that $\alpha_{ij} \triangleright \gamma$. Then, $\sup\{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \triangleright \gamma$, with $a^\mathcal{I} \in \{o_{i_1}, \ldots, o_{i_k}\}^\mathcal{I}$. By construction of $\mathcal{I}_\mathcal{C}$, it holds that $a^{\mathcal{I}_\mathcal{C}} \in \{o_{i_1}, \ldots, o_{i_k}\}^{\mathcal{I}_\mathcal{C}} \iff a^{\mathcal{I}_\mathcal{C}} \in \rho(\{\alpha_1/o_1, \ldots, \alpha_n/o_n\}^{\mathcal{I}_\mathcal{C}}, \triangleright\gamma) \iff \mathcal{I}_\mathcal{C} \models a : \rho(\{\alpha_1/o_1, \ldots, \alpha_n/o_n\}, \triangleright\gamma))) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \{\alpha_1/o_1, \ldots, \alpha_n/o_n\} \triangleright \gamma\rangle)$. The case $\triangleleft\gamma$ is quite straightforward.
   - *at least qualified number restriction*. Assume that $\mathcal{I} \models \langle a : (\geqslant m\ S.C) \geqslant \alpha\rangle$. Then, $\sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}}[(\otimes_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \otimes (\otimes_{j<k}\{b_j \neq b_k\})] \geqslant \alpha$. Note that $(\otimes_{j<k}\{b_j \neq b_k\})$ can be either 0 or 1. If it is 0, then we have that $\sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}}[(\otimes_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \otimes 0] = 0 \geqslant \alpha$, which is not possible because by definition $\alpha \in (0, 1]$. Hence, $(\otimes_{j<k}\{b_j \neq b_k\}) = 1$ and consequently $\sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}}[(\otimes_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \otimes 1] = \sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}}(\otimes_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \geqslant \alpha$. Due to the witnessed model property, there exist m different $b_i \in^{\mathcal{I}_\mathcal{C}}$ such that $\otimes_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}$ and, under minimum t-norm, $S^\mathcal{I}(a^\mathcal{I}, b_i) \geqslant \alpha$ and $C^\mathcal{I}(b_i) \geqslant \alpha$, for $1 \leqslant i \leqslant m$. By induction hypothesis, $(a^{\mathcal{I}_\mathcal{C}}, b_i) \in (\rho(S, \geqslant \alpha))^{\mathcal{I}_\mathcal{C}}$ and $b_i \in (\rho(C, \geqslant \alpha))^{\mathcal{I}_\mathcal{C}}$, for $1 \leqslant i \leqslant m$. By definition of ρ, $a^{\mathcal{I}_\mathcal{C}} \in (\geqslant m\rho(S, \geqslant \alpha).\rho(C, \geqslant \alpha))^{\mathcal{I}_\mathcal{C}} \iff a^{\mathcal{I}_\mathcal{C}} \in \rho(\geqslant m\ S.C, \geqslant \alpha)^{\mathcal{I}_\mathcal{C}} \iff \mathcal{I}_\mathcal{C} \models a : \rho(\geqslant m\ S.C, \geqslant \alpha) \iff \mathcal{I}_\mathcal{C} \models \kappa(\langle a : (\geqslant m\ S.C) \geqslant \alpha\rangle)$. The case $> \beta$ is quite similar. Now assume that $\mathcal{I} \models \langle a : (\geqslant m\ S.C) \leqslant \beta\rangle$. In this case, it follows that $\sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}}$ $(\otimes_{i=1}^m$

[6] It is easy to see that (*i*) implies this condition. To see the equivalence, consider γ' such that $R^\mathcal{I}(a^\mathcal{I}, b) = \gamma'$ and assume that $R^\mathcal{I}(a^\mathcal{I}, b) \geqslant \gamma$ implies $C^\mathcal{I}(b) \geqslant \gamma$. Since $R^\mathcal{I}(a^\mathcal{I}, b) \geqslant \gamma'$ is true, $C^\mathcal{I}(b) \geqslant \gamma'$ holds. Now, it follows that $R^\mathcal{I}(a^\mathcal{I}, b) \leqslant C^\mathcal{I}(b)$, because if $R^\mathcal{I}(a^\mathcal{I}, b) > C^\mathcal{I}(b)$, then $R^\mathcal{I}(a^\mathcal{I}, b) > C^\mathcal{I}(b) \geqslant \gamma'$, which is in contradiction with the assumption that $R^\mathcal{I}(a^\mathcal{I}, b) = \gamma'$.

$\{S^{\mathcal{I}}(a^{\mathcal{I}_c}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \leqslant \beta$. Consequently, there cannot exist m different individuals b_i with $(\otimes_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}_c}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) > \beta$, so we end up with $a^{\mathcal{I}_c} \in (\leqslant m-1 \rho(S, > \beta).\rho(C, > \beta))^{\mathcal{I}_c}$ and finally $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : (\geqslant m\ S.C) \leqslant \beta\rangle)$. The case $< \alpha$ is quite similar.

- *at-most qualified number restriction*. Assume that $\mathcal{I} \models \langle a : (\leqslant n\ S.C) \geqslant \alpha\rangle$. Then, $\inf_{b_1,\dots,b_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \geqslant \alpha$. Note that $(\oplus_{j<k}\{b_j = b_k\})$ can be either 0 or 1, so the result of the Gödel implication is either 0 or 1 and hence $(\leqslant n\ S.C)$ is actually a crisp concept. Since $\alpha > 0$, it follows that $\inf_{b_1,\dots,b_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \geqslant \alpha = 1$. Then, $\forall b_1, \dots, b_{n+1} \in \Delta^{\mathcal{I}}, [(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] = 1$. This is true in two cases: (*i*) $(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) = 0$, so there exist some b_i such that $S^{\mathcal{I}}(x, b_i) = 0$ or $C^{\mathcal{I}}(b_i) = 0$ hold, or (*ii*) $\oplus_{j<k}\{b_j = b_k\} = 0$ holds. This means that there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) > 0$ and $C^{\mathcal{I}}(b_i) > 0$. By induction hypothesis, there do not exist $n+1$ mutually different individuals $b_i \in \Delta^{\mathcal{I}_c}$ such that $S^{\mathcal{I}_c}(a^{\mathcal{I}_c}, b_i) > 0$ and $C^{\mathcal{I}_c}(b_i) > 0$. Hence, $a^{\mathcal{I}_c} \in (\leqslant n\rho(S, > 0).\rho(C, > 0))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : (\leqslant n\rho(S, > 0).\rho(C, > 0)) \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\leqslant n\ S.C, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : (\leqslant n\ S.C) \geqslant \alpha\rangle)$. The case $> \beta$ is quite similar. Now assume $\mathcal{I} \models \langle a : (\leqslant n\ S.C) \leqslant \beta\rangle$. Then, $\inf_{b_1,\dots,b_{n+1} \in \Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \leqslant \beta$. Thanks to the witnessed model property, it follows that there exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) > 0$ and $C^{\mathcal{I}}(b_i) > 0$. By induction hypothesis, there exist $n+1$ mutually different individuals $b_i \in \Delta^{\mathcal{I}_c}$ such that $S^{\mathcal{I}_c}(a^{\mathcal{I}_c}, b_i) > 0$ and $C^{\mathcal{I}_c}(b_i) > 0$. Hence, $a^{\mathcal{I}_c} \in (\geqslant n+1\rho(S, > 0).\ \rho(C, > 0))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : (\geqslant n+1\rho(S, > 0).\rho(C, > 0)) \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\leqslant n\ S.C, \leqslant \beta) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : (\leqslant n\ S.C) \leqslant \beta\rangle)$. The case $< \alpha$ is similar.
- *local reflexivity*. Assume that $\mathcal{I} \models \langle a : \exists S.\mathtt{Self} \rhd \gamma\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $(a^{\mathcal{I}_c}, a^{\mathcal{I}_c}) \in \rho(S, \rhd\gamma)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models (a, a) : \rho(S, \rhd\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \exists S.\mathtt{Self} \rhd \gamma\rangle)$. Now assume that $\mathcal{I} \models \langle a : \exists S.\mathtt{Self} \lhd \gamma\rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $(a^{\mathcal{I}_c}, a^{\mathcal{I}_c}) \in \rho(S, \lhd\gamma)^{\mathcal{I}_c}$. Hence, it follows that $(a^{\mathcal{I}_c}, a^{\mathcal{I}_c}) \notin (\rho(S, \neg\lhd\gamma))^{\mathcal{I}_c} \iff (a^{\mathcal{I}_c}, a^{\mathcal{I}_c}) \in \neg(\rho(S, \neg\lhd\gamma))^{\mathcal{I}_c} \iff a^{\mathcal{I}_c} \in (\rho(\exists S.\mathtt{Self}, \lhd\gamma))^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(\exists S.\mathtt{Self}, \lhd\gamma) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \exists S.\mathtt{Self}, \lhd\gamma\rangle)$.
- *modified concept*. Firstly, let us consider the case of a triangular modifier *mTri* such that $f_{mTri}(x; t_1, t_2, t_3)$. Assume that $\langle \mathcal{I} \models a : mTri(C) \geqslant \alpha\rangle$. Then, it follows that $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geqslant \alpha$. Let $x_1 \in [0, t_2]$ and $x_2 \in [t_2, 1]$ be those numbers such that $f_{left}(x_1; t_1, t_2, t_3) = \alpha$ and $f_{right}(x_2; t_1, t_2, t_3) = \alpha$. There are several options now, depending on the value of α with respect to t_1 and t_3.
  (a) If $(\alpha > t_1)$ and $(\alpha > t_3)$, then $C^{\mathcal{I}}(a^{\mathcal{I}})$ is lower bounded by x_1 (since $f_{left}(x_1; t_1, t_2, t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geqslant \alpha)$ and upper bounded by x_2 (since $f_{right}(x_2; t_1, t_2, t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geqslant \alpha)$. That is, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant x_1$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant x_2$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, \geqslant x_1)^{\mathcal{I}_c}$ and $a^{\mathcal{I}_c} \in \rho(C, \leqslant x_2)^{\mathcal{I}_c}$. It follows that $a^{\mathcal{I}_c} \in \rho(C, \geqslant x_1)^{\mathcal{I}_c} \sqcap \rho(C, \leqslant x_2)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \geqslant x_1) \sqcap \rho(C, \leqslant x_2) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mTri(C) \geqslant \alpha\rangle)$.
  (b) If $(\alpha > t_1)$ and $(\alpha \leqslant t_3)$, then $C^{\mathcal{I}}(a^{\mathcal{I}})$ is lower bounded by x_1 as in the previous case, but x_2 does not introduce an upper bounded now: as noted in Section 5, $f_{mTri}(1) = t_3$, and since $\alpha \leqslant t_3$ and f_{right} is a strictly decreasing function, the possible upper bound for $C^{\mathcal{I}}(a^{\mathcal{I}})$ would be greater than 1, but we already know that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in [0, 1]$. That is, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant x_1$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, \geqslant x_1)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \geqslant x_1) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mTri(C) \geqslant \alpha\rangle)$.
  (c) The case $(\alpha \leqslant t_1)$ and $(\alpha > t_3)$ is similar, but now $C^{\mathcal{I}}(a^{\mathcal{I}})$ is upper bounded by x_2 and not lower bounded. Now, $C^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant x_2$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, \leqslant x_2)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \leqslant x_2) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mTri(C) \geqslant \alpha\rangle)$.
  (d) Finally, in the case $(\alpha \leqslant t_1)$ and $(\alpha \leqslant t_3)$ there are no bounds, so we only now that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in [0, 1]$ and hence we only know that $\top^{\mathcal{I}}(a^{\mathcal{I}})$. By induction hypothesis, $a^{\mathcal{I}_c} \in \top^{\mathcal{I}}_{\mathcal{C}} \iff \mathcal{I}_{\mathcal{C}} \models a : \top \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mTri(C) \geqslant \alpha\rangle)$.

  The other cases $\langle \mathcal{I} \models a : mTri(C) \bowtie \gamma\rangle$ are similar.Now, let us consider the case of a triangular modifier *mLin* such that $f_{mLin}(x; l)$. Assume that $\langle \mathcal{I} \models a : mLin(C) \rhd \gamma\rangle$. Then, it follows that $f_{mLin}(C^{\mathcal{I}}(a^{\mathcal{I}}; l) \rhd \gamma$. Let $x_l \in [0, 1]$ be such that $f_{mLin}(x_l; l) = \gamma$. Then, it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd x_l$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, \rhd x_l)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \rhd x_l) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mLin(C) \rhd \gamma\rangle)$. The case $\langle \mathcal{I} \models a : mLin(C) \lhd \gamma\rangle$ is similar, but now it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd x_l$ so we end up with $\mathcal{I}_{\mathcal{C}} \models a : \rho(C, \lhd x_l) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : mLin(C) \lhd \gamma\rangle)$.
- *Cut concept*. Assume that $\langle \mathcal{I} \models a : [C \geqslant \alpha] \rhd \gamma\rangle$. Then, it follows that $([C \geqslant \alpha])^{\mathcal{I}}(a^{\mathcal{I}}) = 1$, which is the case if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant \alpha$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, \geqslant \alpha)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, \geqslant \alpha) \iff \mathcal{I}_{\mathcal{C}} \models a : \rho([C \geqslant \alpha], \rhd\gamma)$. Now assume that $\langle \mathcal{I} \models a : [C \geqslant \alpha] \lhd \gamma\rangle$. Then, it follows that $([C \geqslant \alpha])^{\mathcal{I}}(a^{\mathcal{I}}) = 0$, which is the case if $C^{\mathcal{I}}(a^{\mathcal{I}}) < \alpha$. By induction hypothesis, $a^{\mathcal{I}_c} \in \rho(C, < \alpha)^{\mathcal{I}_c} \iff \mathcal{I}_{\mathcal{C}} \models a : \rho(C, < \alpha) \iff \mathcal{I}_{\mathcal{C}} \models a : \rho([C \geqslant \alpha], \lhd\gamma)$.

5. τ is a fuzzy GCI. Assume that $\mathcal{I} \models \langle C \sqsubseteq D \geqslant \alpha\rangle$. Then, $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geqslant \alpha$. Hence, for an arbitrary individual $x \in \Delta^{\mathcal{I}}$ it follows that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geqslant \alpha$ and hence one of the following conditions holds (*i*) $C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(x)$ (which makes the Gödel implication equal to $1 \geqslant \alpha$), or (*ii*) $D^{\mathcal{I}}(x) \geqslant \alpha$ (which makes the Gödel implication take a value $\geqslant \alpha$). Note that the former condition is equivalent to: $C^{\mathcal{I}}(x) \rhd \gamma$ implies $D^{\mathcal{I}}(x) \rhd \gamma$ for every $\gamma \in \mathcal{TV}$. The latter condition allows us to restrict to those $\gamma \in \mathcal{TV}$ such that $\gamma \leqslant \alpha$. By induction hypothesis, it follows that $x^{\mathcal{I}_c} \in (\rho(C, \geqslant \gamma))^{\mathcal{I}_c}$ implies $x^{\mathcal{I}_c} \in (\rho(D, \geqslant \gamma))^{\mathcal{I}_c}$ or $x^{\mathcal{I}_c} \in (\rho(D, \geqslant \alpha))^{\mathcal{I}_c}$, for an arbitrary $x \in \Delta^{\mathcal{I}_c}$. Consequently, it follows that $\mathcal{I}_{\mathcal{C}} \models \bigcup_{\gamma \in \mathcal{TV} \setminus \{0\} | \gamma \leqslant \alpha}\{\rho(C, \geqslant \gamma) \sqsubseteq \rho(D, \geqslant \gamma)\} \bigcup_{\gamma \in \mathcal{TV} | \gamma < \alpha}\{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\} \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\langle C \sqsubseteq D \geqslant \alpha\rangle)$. The case for $> \beta$ is quite similar.
6. τ is a fuzzy RIA. Assume that $\mathcal{I} \models \langle R_1 \dots R_n \sqsubseteq R \rhd \gamma\rangle$. The case is similar to the previous one, with the difference that there appears a minimum i.e., $\min\{R_1^{\mathcal{I}}(y_1, y_2), \dots, R_n^{\mathcal{I}}(y_n, y_{n+1})\} \leqslant \{R^{\mathcal{I}}(y_1, y_{n+1})$. As a consequence, the left side of the crisp RIAs will contain $\rho(R_1, \rhd\gamma) \dots \rho(R_n, \rhd\gamma)$ in the left side, instead of $\rho(C, \rhd\gamma)$.
7. τ is a role disjoint axiom. Assume that $\mathcal{I} \models \mathtt{dis}(S_1, S_2)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = 0$ or $S_2^{\mathcal{I}}(x, y) = 0$. By induction hypothesis, $\forall x, y \in \Delta^{\mathcal{I}_c}, (x, y) \in (\rho(S_1, \leqslant 0))^{\mathcal{I}_c}$ or $(x, y) \in (\rho(S_2, \leqslant 0))^{\mathcal{I}_c} \iff \forall x, y \in \Delta^{\mathcal{I}_c}, (x, y) \notin (\rho(S_1, > 0))^{\mathcal{I}_c}$ or $(x, y) \notin (\rho(S_2, > 0))^{\mathcal{I}_c} \iff (\rho(S_1, > 0))^{\mathcal{I}_c} \cap (\rho(S_2, > 0))^{\mathcal{I}_c} = \emptyset \iff \mathcal{I}_{\mathcal{C}} \models (dis(\rho(S_1, > 0), \rho(S_2, > 0))) \iff \mathcal{I}_{\mathcal{C}} \models \kappa(\mathtt{dis}(S_1, S_2))$.

8. τ is a reflexive role axiom. Assume that $\mathcal{I} \models \mathtt{ref}(R)$. Then, $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$. By induction hypothesis, $\forall x \in \Delta^{\mathcal{I}_C}, (x,x) \in (\rho(R, \geqslant 1))^{\mathcal{I}_C} \iff \forall x \in \Delta^{\mathcal{I}_C}, \mathcal{I}_C \models (x,x) : \rho(R, \geqslant 1) \iff \mathcal{I}_C \models \kappa(\mathtt{ref}(R))$.
9. τ is an asymmetry role axiom. Assume that $\mathcal{I} \models \mathtt{asy}(S)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$. By induction hypothesis, $\forall x, y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S, > 0))^{\mathcal{I}_C}$ then $(y,x) \in (\rho(S, \leqslant 0))^{\mathcal{I}_C} \iff \forall x, y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S, > 0))^{\mathcal{I}_C}$ then $(y,x) \notin (\rho(S, > 0))^{\mathcal{I}_C}$. Consequently, $\mathcal{I}_C \models \kappa(\mathtt{asy}(\rho(S, > 0)))$.

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If $\kappa(\mathcal{K})$ is satisfiable, it is not possible (due to the axioms in $T(\mathcal{TV})$) to have an individual a such that $a^{\mathcal{I}_C} \in (A_{\triangleright\gamma_1})^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \notin (A_{\triangleright\gamma_2})^{\mathcal{I}_C}$ with $\gamma_2 < \gamma_1$, so for every individual a we can compute the maximum value α such that $a : A_{\geqslant\alpha}$ holds, or the maximum value β such that $a : A_{>\beta}$ holds, and use these values in the construction of the fuzzy interpretation. The case for roles is similar. □

A.6. Proof of Theorem 4.4

Trivial from the following observations:

- Every axiom is reduced to a combination of new crisp elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- τ does not introduce atomic concepts, atomic abstract roles, concrete roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others. □

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